3. Physical fundamentals of mechanics

3.1. Introduction to Mechanics

3.1.1 Subject of mechanics. The concept of the mechanical motion. Body of reference and frame of reference

Mechanics is the chapter of physics that studies the movement of the matter which consists in the simple displacement one of body relative another body (or system of them) accepted as unmoving in the problem that there is under consideration accepted as unmoving. The body which in the considered problem accepted as unmoving is called *the body of reference*. The body which in the considered problem is accepted as unmoving and the devices and attachments for time intervals and space distances measuring as well connected with the giving body form *the frame reference.*

3.1.2. Stages of mechanical development. Classical, relativistic, and quantum mechanics

In own development mechanics passed so stages:

- classical non-relativistic mechanics (or Newtonian – Galilean mechanics);
- classical relativistic mechanics;
- quantum non-relativistic mechanics;
- quantum relativistic mechanics.

Near we will give review of this stage. We will notice the first stage relates to description of body motion on the base of Newton laws. Later on it was found that these laws have the restricted domain of applicability. On one hand they turned inapplicable to description of bodies moving with a speed v close to the light speed $c = 3 \cdot 10^8 m/\sec$, that is

$$v \leq c \tag{A}$$

On the other hand it was established that all material objects possess by mutual nature namely particle-wave one. More exactly some wave is connected with each material object. This wave is called de Broglie one with wavelength

$$\lambda = h/p, \tag{B}$$

where $h = 6{,}62 \cdot 10^{-34} Joul\square\square e{\cdot}sec$ is a Planck's constant, $p = \dfrac{mv}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ is object momentum, m is the object mass and v is its speed. The wave properties of somebody become essential if it locates in linear size area a which satisfies to the relation $a \sim \lambda$.

One may ignore bodies wave properties if the condition

$$a >> \lambda \tag{C}$$

takes place for linear sizes of their localization area a. In this case mechanics is called *classical*. In the case when the inequality

$$v << c \tag{D}$$

takes place mechanics is called *non-relativistic.* When both of conditions (C) and (D) take place we have the *classical non-relativistic* mechanics or *Newtonian* –

Galilean mechanics (it is called shortly the *Newtonian* mechanics).

As it is seemed from the relations (B) – (D) *Newtonian mechanics is used for slow motion macroscopic bodies describing*.

If the conditions (A) and (C) take place mechanics is called *classical relativistic.* In this case the relativistic effects are taken into account but the wave properties occurrence of the somebody is ignored as before. Conceptions of this mechanics are used in the describing of the electromagnetic waves propagation, some radioactive radiations types propagation etc. In common case one may say using of the classical relativistic mechanics is acceptable when the body has motion velocity which satisfies the inequality (A) and when spatial area of its localization is large enough (the inequality (C) is satisfied as it was noticed above).

In nature the case may be realized when inequality

$$a << \lambda \qquad \text{(E)}$$

takes place and the inequality (D) is valid also. Then we say about *nonrelativistic quantum mechanics,* which is used for describing the objects behaviour when one may ignore its relativistic properties (slow motion when the inequality (D) satisfies) but it is necessary to take into account their wave properties. The application objects of this mechanics are electrons of light atoms, electrons from the outer atomic shells in heavy atoms, electrons in the crystal lattice, etc. Domain of applicability of this mechanics is wide enough; particularly on its base heat, electrical and magnetic properties of solid are explained.

If the inequalities (A), (D) we have the *relativistic quantum mechanics*. In this case at describing the objects behaviour one has to take into account as their relativistic properties connected with large motion velocity so their wave ones. As examples of the objects which for their describing require the relativistic quantum mechanics we will point out the electrons from the interior atomic shells in heavy atoms, atomic nuclei, microparticles in the accelerators etc. Besides consequent theory describing the objects behaviour with taking into account both their relativistic properties and the quantum ones simultaneously does not originated up to date.

3.1.3. Newtonian and relativistic conceptions of space and time

Mechanical motion consists in simple displacement somebody in the space in the course of time. The question arises about nature space and time, and about their properties.

By space we will mean the philosophical category which is the form of the matter existence and which is connected with establishing sizes of the real subjects and their mutual displacement. By time we will mean the philosophical category also establishing the order of events in space and their duration. It is necessary to distinguish in the approach to solving of the problem in Newtonian and relativistic mechanics. In Newtonian mechanics space and time have the absolute character: space-time relations between objects and events do not depend on the viewer point of view. Besides relations mention above are independent ones from others. Finally in Newtonian mechanics space and time possess by properties of uniformity and in additional to this space possesses by the property of isotropy. Space uniformity means that in it there are no the pointed out points. In another words if one wants to select the origin of coordinate system there are no points which would be the most acceptable for this purpose. One may say the same in relative to time: on the time axis there are no the acceptable points for the origin of time reference selection.

By passing to the relativistic conceptions of space and time it should kept in mind that these conceptions have some differences in special relativity and in general one. Let us consider the space and time properties which are common in both cases. First of all in this case space as well as time lose its absolute character and become relative concepts. Here the sizes of the objects and duration of the events and even their order depend on the viewer point of view. In this case instead of individual matter existence forms, space and time there is a single form of space – time. Whereas in Newtonian mechanics the event is characterized by space coordinates and by moment of time separately in relativistic mechanics it is characterized by the point in 4-dimensional space-time. In special relativity unified space-time possesses properties of uniformity and isotropy.

Situation in relativistic mechanics is manifestly described by the great A. Einstein's teacher G. Minkovsky: «***The views of space-time, which I'm going to develop in front of you, have grown on the basis of experimental physics. That is their strength. They will lead to drastic consequences. Now the space itself, as well as time itself fully go into the realm of shadows, and only a union of both of these concepts retains an independent existence».***

Concerning general relativity we note in this case in comparison with the special one the 4-dimensional space-time properties of uniformity and isotropy are lost. Its properties occur closely connected with mass distribution in this space. This circumstance is good illustrated by Einstein's words: «***Substance indicates the space how to curve and space indicates matter how to move».***

3.1.4. Structure of mechanics as chapter of physics course

As chapter of physics course mechanic in one's turn consists of the following chapters:

- kinematics;
- dynamics;
- statics.

We will characterize these chapters. Kinematics is a chapter of mechanics which studies bodies motion without taking into account factors effecting on the motion character. Because on the motion character of the body its interaction with others bodies effects one may say that kinematics studies bodies motion without taking into account their interaction with others bodies. This chapter has the descriptive character.

Dynamics is a chapter of mechanics which studies bodies motion with taking into account factors effecting on the motion character. Dynamics is the «heart» of mechanics. It describes bodies motion with taking into account interaction between bodies and therefore describing of the mechanical processes on the base of dynamics is more complete than on the kinematics base.

Statics is a chapter of mechanics which studies the conditions of bodies equilibrium being under the forces acting. This chapter often is considered as a particular case of dynamics. In this part of book we pass to learn of kinematics.

3.2. Elements of kinematics

Learning task

- become familiar with the basic kinematics concepts and learn to use them in solving the specific problems.

Glossary

Material point (particle) – is a physical body possessing by the mass but by the form and by the sizes of which it is possible to neglect in the problem under consideration.

Radius-vector – is a vector which is directed from the coordinates origin into the point where the particle is.

Displacement vector – is a radius-vector difference of two points of the particle trajectory.

Velocity vector – is a vector characterizing the speed of the particle radius-vector changing.

Acceleration vector – is a vector characterizing the speed of the particle velocity vector changing.

Tangential acceleration vector – is a total acceleration component which answerable for the changing of the velocity vector modulus.

Normal acceleration vector – is a total acceleration component which answerable for the changing of the velocity vector direction.

Circle of the curvature of the curve at the point – is a circle having a common point with given curve and touching it at two other closely located points.

Curvature radius of the curve at given point – is the radius of the circle of the curvature of the curve at the point.

Traversed path – is a distance between two points on the particle trajectory at which the particle becomes.

Vector of the rotation angle (angular displacement) – is a vector which is on the rotation axis, modulus of which is equal to angular rotation magnitude, and which has such orientation that from its end considered rotation seems to be occuring anticlockwise

Vector of the angular velocity – is a vector characterizing the speed of the particle angular coordinates changing.

Vector of the angular acceleration – is a vector characterizing the speed of the particle angular velocity vector changing.

Rigid body – is a particles system in which the distances between any couple of them are assumed to be constant.

Number of freedom degrees of a mechanical system – is a quantity of the independent variables which is termed the position of the system in space.

Generalized coordinates – are the independent parameters describing the position of a mechanical system in space, number of which is equal to the number of freedom degrees of the system.

From the beginning we will introduce the concepts of material point: *as material point we will suppose the physical body possessing by the mass but by the form and by the sizes of which it is possible to neglect in the problem under consideration.* For reasons of convenience instead of the term «material point» further we will use the term «particle». Nearer we will introduce the main kinematic concepts.

3.2.1. Radius-vector

Radius-vector is a vector directed from the coordinates origin into the point where the particle is situated (see a figure 3.1). This vector is characterized by its coordinates or by its projections. It may be represented as a sum of vectors which are called components of this vector. These components define as follows.

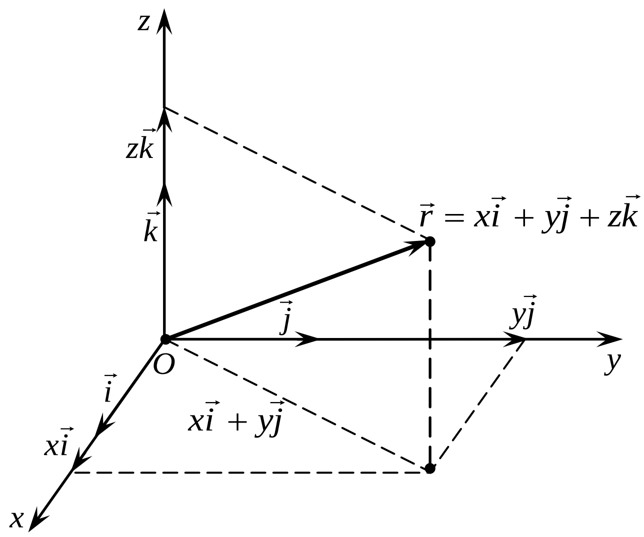

Figure 3.1. Radius-vector definition

Let us define the unit vectors $\vec{i}, \vec{j}, \vec{k}$ that have the unit length. These vectors are called by *directions orts*. Then we have

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}\,.$$

If the particle moves its coordinates x, y, z become the function of time, i.e.

$x \to x(t); y \to y(t); z \to z(t);$ then

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\,. \quad (3.1)$$

In mechanics coordinates of the particle are accepted as the parameters, which define its mechanical state. Then, one of the major problem in kinematics is the determination of the particle radius-vector in any point of time if it is known in the point accepted as the original. In order to solve this problem we will introduce some new concepts defined lower.

3.2.2. A displacement vector

Let us consider the motion of the particle along the trajectory. We will show the particle position on the trajectory in the time point t and in the point $t + \Delta t$. Then the radius-vectors of the particle defined by the equality (3.1) are $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ and for difference between $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ we have

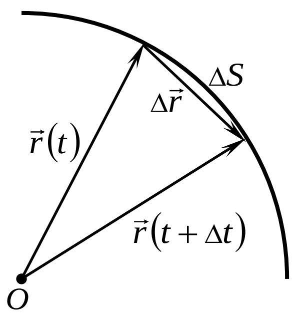

Figure 3.2. To the displacement vector definition

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t). \quad (3.2)$$

This vector is called by *displacement vector* (see figure 3.2). We will emphasize that any assumptions about smallness Δt are not made here.

3.2.3. A velocity vector

An interest of a change radius-vector speed may be for somebody. This speed is characterized by the quantity called by the *velocity vector*. We will define a quantity called by the *average velocity for time interval of* Δt. This quantity is defined as follows:

$$\langle \vec{v} \rangle_{\Delta t} = \frac{\Delta \vec{r}}{\Delta t}, \qquad (3.3)$$

where for $\Delta \vec{r}$ the expression (3.2) is used. Further we will define an *instantaneous velocity vector* or a velocity vector in point of time t. Using the equality (3.3) we will define an instantaneous velocity vector by the equation

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}. \qquad (3.4)$$

The expression $\frac{d\vec{r}}{dt}$ denotes that *the instantaneous velocity is the first derivative of the radius-vector in respect to time*.

It is of interest to determine the direction of the instantaneous velocity vector. For this purpose from the beginning we will establish the direction of the average velocity and for solving given problem we will have recourse to the relation (3.3) and figure 3.2. Then we see that vector $\Delta \vec{r}$ lies on the secant line of the particle trajectory and the directions of vectors $\Delta \vec{r}$ and $\langle \vec{v} \rangle_{\Delta t}$ coincide. When the limiting process $\Delta t \to 0$ occurs both points of the vector $\langle \vec{v} \rangle_{\Delta t}$ intersecting with the trajectory approach each to other and the secant line of the trajectory takes the position of tangent line. Therefore one may conclude that *the direction of the instantaneous velocity vector in some point of the particle trajectory coincides with the tangent line direction in the same point of trajectory*.

For the direction of the instantaneous velocity vector it is possible to trace in the next experiment. We will take the grindstone and give it to rotation. After that in a some point of the grindstone we will touch by the cusp of the file. This gives rise to sparks that come off the stone and the direction of the velocity of which in the separation moment coincides with the direction of the stone point linear velocity from which came off a spark. The experiment described above is illustrated by the figure 3.3.

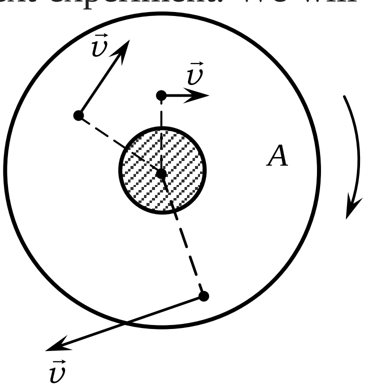

Figure 3.3. Experimental illustration of the instantaneous velocity vector direction: *when the grinstone A rotates and the cups touches at the some point from this point the spark comes off at the same direction which has the grinstone point of touching by the cups*

The expression (3.4) defining the $\vec{v}(t)$ in one's turn is equivalent to three scalar equalities:

$$v_x(t) = \frac{dx}{dt}; v_y(t) = \frac{dy}{dt}; v_z(t) = \frac{dz}{dt}. \qquad (3.5)$$

With the help of the expressions (3.4), (3.5) for modulus of the instantaneous velocity vector we can write:

$$v(t)=\sqrt{v_x^2(t)+v_y^2(t)+v_z^2(t)}. \tag{3.6}$$

Velocity vector also may be presented in the form

$$\vec{v}=v\cdot\vec{e}_v, \tag{3.7}$$

where v is the modulus of vector and $\vec{e}_v$ is a vector of unit length which indicates the direction of the velocity vector; it is called by velocity ort. This form of velocity vector is very convenient for characteristic of its changing.

3.2.4. Traversed path

Here we will find the path traversed by the particle. For this purpose we will have recourse to the relation (3.4). This relation allows us writing

$$v(t)=|\vec{v}(t)|=\lim_{\Delta t\to 0}\frac{|\Delta\vec{r}|}{\Delta t}. \tag{3.8}$$

Using smallness of the quantity $|\Delta\vec{r}|$ and as it may be shown from the figure 3.2 we will write that

$$|\Delta\vec{r}|\approx\Delta S, \tag{3.9}$$

where ΔS is the length of the path traversed by the particle over space of time Δt accepted near fixed moment t. Then relations (3.8) and (3.9) allow us writing

$$v(t)=|\vec{v}(t)|=\lim_{\Delta t\to 0}\frac{\Delta S}{\Delta t}=\frac{dS}{dt}. \tag{3.10}$$

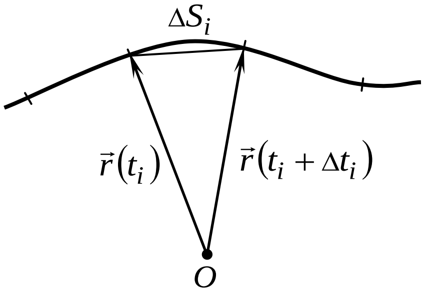

Figure 3.4. To definition of the traversed path of the body

It is follows from (3.10) that we may write

$$\Delta S\approx v(t)\cdot\Delta t; \tag{3.11}$$

here we disregarded by the changing of modulus $v(t)$ within the interval $t, t+\Delta t$.

The equality (3.11) can be considered as relating to the element of the particle trajectory with number «*i*» on which the trajectory is separated (figure 3.4). Then as it follows from (3.11) we can write

$$\Delta S_i\approx v(t_i)\cdot\Delta t_i.$$

For total path traversed by the particle from the moment of time t_1 till the moment t_2 we obtain

$$S=\lim_{\Delta t\to 0}\sum_{i=1}^{i=n}v(t_i)\cdot\Delta t_i=\int_{t_1}^{t_2}v(t)\cdot dt. \tag{3.12}$$

Thereby we see that *for calculation of the traversed path it is necessary to know the modulus velocity vector changing law over the time.*

3.2.5. An acceleration vector

3.2.5.1. Definition of the acceleration vector

For many mechanical problems it is necessary to know the seed of the velocity vector changing. For this purpose the quantity is introduced which named by the *acceleration vector*. From the beginning we will define the average acceleration for the tine interval of Δt:

$$\langle \vec{a} \rangle_{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}, \tag{3.13}$$

where $\Delta \vec{v}$ is changing of vector $\vec{v}$ for tine interval of Δt. Then using the relation (3.13) we may define an instantaneous acceleration by the equality

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}. \tag{3.14}$$

Thereby *the acceleration vector is the first derivative of the velocity vector in respect to time*, or using definition of the instantaneous velocity (3.4) and equality (3.14) we have

$$\vec{a}(t) = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}, \tag{3.15}$$

that denotes that *the instantaneous acceleration is the second derivative of the radius-vector in respect to time*.

In the case of the acceleration vector we can write the same expressions for its projections as it was made for the velocity vector (relations (3.5), (3.6)). Then we obtain

$$a_x(t) = \frac{dv_x}{dt}; a_y(t) = \frac{dv_y}{dt}; a_z(t) = \frac{dv_z}{dt}, \tag{3.16}$$

and for modulus of the acceleration vector we have

$$a(t) = \sqrt{a_x^2(t) + a_y^2(t) + a_z^2(t)}. \tag{3.17}$$

3.2.5.2. A tangential and normal components of acceleration

We will consider the accelerator vector of at greater length. For this purpose we will use the relation (3.7) expressing the velocity vector through its modulus and its ort. After inserting the equality (3.7) into relation (3.14) we obtain

$$\vec{a}(t) = \frac{dv}{dt} \cdot \vec{e}_v + v \cdot \frac{d\vec{e}_v}{dt}. \tag{3.18}$$

Thereby we see that in the common case the acceleration vector is represented by the sum of two terms. The first of them

$$\vec{a}_\tau = \frac{dv}{dt} \cdot \vec{e}_v \tag{3.19}$$

is called by *the tangential acceleration*. *This vector lies on the same line as the velocity vector and it is answerable for the changing of the velocity vector* *modulus*. One can see from (3.19), that if the velocity vector modulus increases with over time $\left(\frac{dv}{dt} > 0\right)$ the tangential acceleration vector coincides with the velocity vector by the direction, and if over time the velocity vector modulus decreasing $\left(\frac{dv}{dt} < 0\right)$ takes place both vectors $\vec{a}_\tau$ and $\vec{v}$ have the opposite directions. Therefore for modulus of the tangential acceleration vector it is possible to write

$$a_\tau = \left|\frac{dv}{dt}\right| = \left|\frac{\vec{v}\cdot\vec{a}}{v}\right| = \frac{\left|v_x \cdot \frac{dv_x}{dt} + v_y \cdot \frac{dv_y}{dt} + v_z \cdot \frac{dv_z}{dt}\right|}{\sqrt{v_x^2 + v_y^2 + v_z^2}}. \tag{3.20}$$

The second term in sum (3.18) is connected with the *direction changing of the vector velocity*, and it is called by the *normal acceleration*.

Further we will define the relative orientation of the vectors mentioned above. We notice that vectors $\vec{a}_\tau$ and $\vec{e}_v$ are parallel (or are antiparallel) one to another and vector $\vec{a}_n$ has the same properties in respect to the vector $d\vec{v}/dt$. Let us consider scalar product

$$\vec{e}_v \cdot \vec{e}_v = \vec{e}_v^{\,2} = 1 \tag{3.21}$$

(we used the fact that vector $\vec{e}_v$ has unit length). Now we take the derivative

$$\frac{d\vec{e}_v^{\,2}}{dt} = 2\vec{e}_v \cdot \frac{d\vec{e}_v}{dt}.$$

From the equality (3.21) it follows that

$$\vec{e}_v \cdot \frac{d\vec{e}_v}{dt} = 0. \tag{3.22}$$

The relation (3.22) shows that vectors $\vec{e}_v$ and $\frac{d\vec{e}_v}{dt}$ are mutual orthogonal. In one's turn from the last assertion it is follows that vectors $\vec{a}_\tau$ and $\vec{a}_n$ are mutual orthogonal also, i.e.

$$\vec{a}_\tau \perp \vec{a}_n. \tag{3.23}$$

Thereby from the relation (3.18) we can write

$$\vec{a} = \vec{a}_\tau + \vec{a}_n, \tag{3.24}$$

where vector $\vec{a}$ is called by the *vector of total acceleration*. From the relations (3.18) and (3.19) it follows that the modulus of total acceleration vector may be defined by the equality

$$a = \sqrt{a_\tau^2 + a_n^2}. \tag{3.25}$$

Formula (3.25) is completely equivalent to (3.17) for calculation of total acceleration.

Now we are ready to solving the major problem of kinematics noticed at the end of paragraph **3.2.1.** We will consider the right part of the expression (3.4) as vulgar fraction; then we can write

$$d\vec{r} = \vec{v}(t)\cdot dt. \tag{3.26}$$

In written equality $d\vec{r}$ is a small displacement which particle suffers during time interval dt. For finding of the finite displacement vector we have to integrate the equality (3.26) that is

$$\int_{\vec{r}(t_o)}^{\vec{r}(t)} d\vec{r} = \int_{t_o}^{t} \vec{v}(t)\cdot dt,$$

where t is the present point of time, t_0 is the point of time which accepted as original, $\vec{r}(t)$ is the radius-vector of the particle at the present point of time, and $\vec{r}(t_0)$ is the radius-vector of the particle at the point of time which is accepted as original.

After making of the integration and performing simple transformation we finally obtain

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(t)\cdot dt. \tag{3.27}$$

Thereby we see that for determination particle radius-vector at any point of time one must know it at some point accepted as original and also the low of the velocity vector changing $\vec{v}(t)$ over the time. As it follows from relations (3.14), (3.26) and (3.27) in one's *turn for finding this law we must know the total acceleration vector changing law over the time*.

In connection with said above *the question arises about of higher derivatives in respect to time of radius-vector knowledge for calculation of this vector. The reply is given in dynamics and consists in the following. Nature is organized so that knowledge of higher derivatives of radius-vector in respect to time is not necessary and for solving stated problem restriction by the first two derivatives in respect to time is quite enough.*

3.2.5.3. Some geometrical information

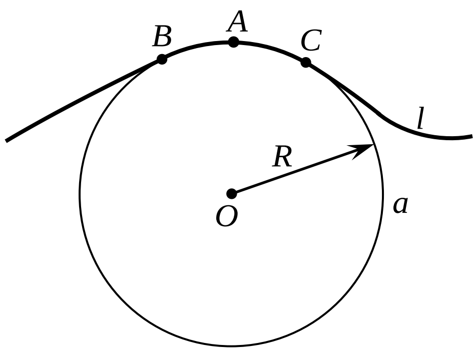

Figure 3.5. Elements of planar curve geometry:
a – the circle of the curve curvature,
O – the curve curvature centre at the point *A*,
R – the radius of the curve curvature at the point *A*.

For fullness of the considering it remains to find to what equal the modulus of the normal acceleration vector. For this purpose it is necessary to introduce some geometrical concepts. We will introduce some geometric definitions and for this purpose we will consider some curve *l* (figure 3.5). In geometry there is the next important statement: *one can represent almost any curve as a set of a circle arcs with different radii and with* *different centers locations*. If one chooses the point A (figure 3.5) and the points B and C in its neighbourhood he can draw one and only one circle through these points. This circle is named *by the circle of the curve curvature in the point A, the point O is named by the curve curvature centre and the radius of this circle R is named by the curvature radius of the curve in given point*[1]. Last assertions allow us to consider particle motion along the circle instead of its motion along the arbitrary curve.

In addition to mention above we will show how the curvature radius of the planar curve may be evaluated. On the figure 3.6a is represented rotating of the unit tangential vector $\vec{e}$ of the some planar curve which in consequence of the assertions expressed above may be considered as rotating along the circle. As a measure of the curve curvature at the point we will take

$$k = \lim_{\Delta S \to 0} \frac{\Delta\varphi}{\Delta S} = \frac{d\varphi}{dS}, \tag{3.28}$$

where $\Delta\varphi$ is a rotating angle of unit vector $\vec{e}$ when this vector displaces to the point which is at a distance ΔS from the first of them.

We can establish connection between the curvature and radius of the curvature at the point. Taking in the account that the angle $\Delta\varphi$ is central we may write $\Delta\varphi = \frac{\Delta S}{R}$, whereas

$$k = \lim_{\Delta S \to 0} \frac{\Delta\varphi}{\Delta S} = \frac{d\varphi}{dS} = \frac{1}{R}. \tag{3.29}$$

Further we will use the relation

$$ds^2 = dx^2 + dy^2, \tag{3.30}$$

which follows from the figure 3.6b. Using the relations (3.28) – (3.30) we write

$$k = \frac{d\varphi}{\sqrt{dx^2 + dy^2}} = \frac{d\varphi}{dx} \cdot \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}. \tag{3.31}$$

From the definition of the geometrical meaning of derivative (figure 3.6b) we have

$$\frac{dy}{dx} = y'(x) = \operatorname{tg}\varphi(x)$$

and inversion of the function $\operatorname{tg}\varphi(x)$ gives

$$\varphi(x) = \operatorname{arctg} y'(x),$$

whereas

$$\frac{d\varphi}{dx} = \frac{y''(x)}{1 + y'^2(x)}. \tag{3.32}$$

After substitution of the relation (3.32) into (3.31) for curve curvature we obtain

[1] Work of such drafting instrument as French curve is based on these assertions.

$$k = \frac{y''(x)}{\left(1 + y'^2(x)\right)^{3/2}}. \tag{3.33}$$

With the help of the relation (3.29) found expression allows us to define the curvature radius at the fixed point of the curve.

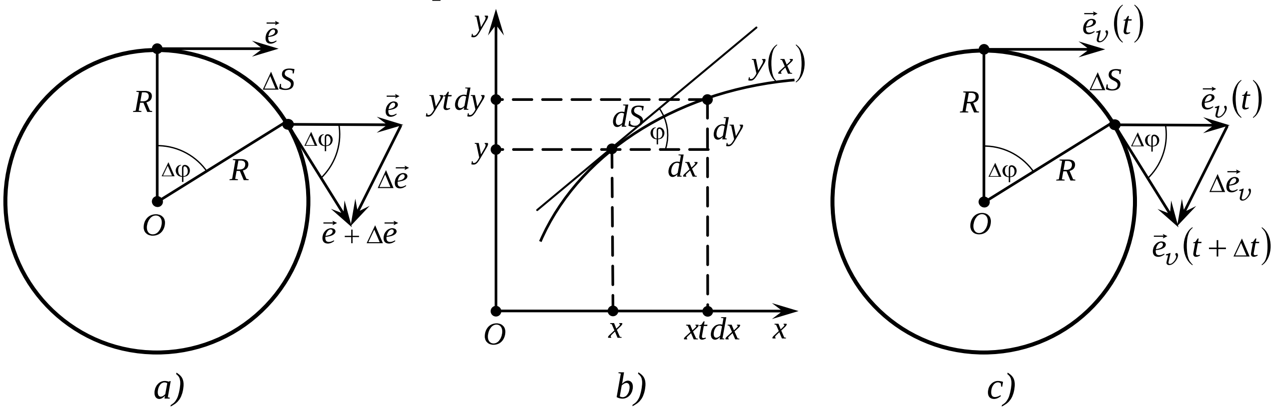

Figure 3.6. To derivation of the formula (3.29) for the curve curvature at the point and for the formula (3.36) for the normal acceleration:

a – to derivation of the formula (3.29); *b* – to derivation of the formula (3.33) for the curve curvature at the point; *c* – to derivation of the formulas for the curve curvature at the given point and modulus of the normal acceleration vector.

3.2.5.4. Vector of the normal acceleration; calculation of its modulus and its direction definition

For solving of the problem formulated in the title we will address to the figure 3.6c. In this figure vector $\vec{e}_v(t)$ shows the velocity direction in the moment of time t and vector $\vec{e}_v(t+\Delta t)$ shows the vector direction in the moment of time $t+\Delta t$. If one wants to definite vector $\vec{a}_n$ he must find the difference $\Delta\vec{e}_v = \vec{e}_v(t+\Delta t) - \vec{e}_v(t)$. For finding vector of the normal acceleration $\vec{a}_n$ using relationship (3.18) we can write

$$\vec{a}_n = v \lim_{\Delta t \to 0} \frac{\Delta\vec{e}_v}{\Delta t} = v \frac{d\vec{e}_v}{dt}.$$

For modulus of $\vec{a}_n$ also write

$$a_n = v \lim_{\Delta t \to 0} \frac{\left|\Delta\vec{e}_v\right|}{\Delta t}. \tag{3.34}$$

For finding modulus $\left|\Delta\vec{e}_v\right|$ we will transfer vector $\vec{e}_v(t)$ parallel to itself in the point where vector $\vec{e}_v(t+\Delta t)$ is situated. Then vector modulus $\left|\Delta\vec{e}_v\right|$ may be found from the isosceles triangle formed by the vectors $\vec{e}_v(t)$, $\vec{e}_v(t+\Delta t)$ and $\Delta\vec{e}_v$ (see figure 3.6c). We will use the theorem about the angles formed by the mutual perpendicular sides (these angles are shown in the figure 3.6c), whereupon we have

$$\left|\Delta\vec{e}_v\right| = 2\left|\vec{e}_v\right| \cdot \sin\frac{\Delta\varphi}{2}$$

and further using by the smallness of the angle $\Delta\varphi$ we obtain

$$\left|\Delta\vec{e}_v\right| = 2 \cdot \frac{\Delta\varphi}{2} \approx \Delta\varphi. \tag{3.35}$$

Then from the relationships (3.34) and (3.35) it follows

$$a_n = v \lim_{\Delta t \to 0} \frac{\left|\Delta\vec{e}_v\right|}{\Delta t} = v \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t}.$$

Further we will multiply the right side of the obtained equality by R (circle radius) and divide it by the same quantity:

$$a_n = \frac{v}{R} \lim_{\Delta t \to 0} \frac{\Delta\varphi \cdot R}{\Delta t}.$$

The numerator of the obtained fraction $\Delta S = \Delta\varphi \cdot R$ is the circle arc length on which the central angle is leaned. In consequence of saying and taking into account relation (3.10) we have

$$a_n = \frac{v}{R} \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \frac{v^2}{R}. \tag{3.36}$$

Written formula is the final result for which obtaining we strived.

Concerning direction of the vector $\vec{a}_n$ we will say the next. As it follows from the expression (3.15) the vectors $\vec{a}_n, \vec{a}_\tau$ are mutually transverse one to another. In additional to that from the figure (3.6c) one may see that when the angle $\Delta\varphi$ tends to zero vector $\Delta\vec{e}_v$ defining *orientation of vector* $\vec{a}_n$ *tends to the acceptance of the circle radius position and is directed to the centre of the circle. Therefore vector* $\vec{a}_n$ *is often called as the centripetal acceleration.*

3.2.6 Rotational motion kinematics

3.2.6.1 Rotational motion kinematics of the material point. Vector of the rotation angle

Lower it is considered the rotation around the fixed axis**.** Let us in the arbitrary point of time the particle is in the point *A* (figure 3.7a), and over the space of time Δt it passes to the point *B*; so that it rotates on the angle $\Delta\varphi$. One may interest in the rotation direction in the analogue with the case of the translational motion when we designate not only a magnitude but also the direction of the motion. In order to do it, it is introduced the quantity which is called by *the rotation angle vector or by the vector of the angular displacement* $\Delta\vec{\varphi}$. *This vector is defined as a vector which there is on the rotation axis, modulus of which* $\Delta\varphi$ *is equal to angular rotation magnitude, and which has such orientation that from its* *end considered rotation seems to be occurring anticlockwise*. The meaning of this value is illustrated by the figure 3.7a.

It should be remembered that vectorial character one may assign to the small angle of rotation only; in particular if we are dealing with angles of arbitrary magnitude they do not satisfy to rules of the vectors summation and therefore it is impossible to consider as a vector an angle of arbitrary magnitude.

3.2.6.2 Vectors of the angular velocity and of angular acceleration. By complete analogy with the case of the translational motion we introduce the vector which characterizes a speed of the particle rotation and its direction. This vector is defined by the relation

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\varphi}}{\Delta t} = \frac{d\vec{\varphi}}{dt}; \qquad (3.37)$$

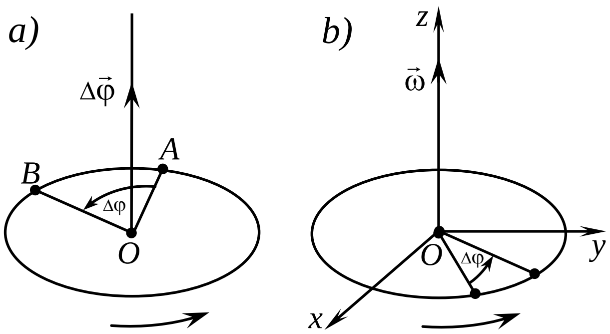

Figure 3.7. To definition of the vectors angular replacement and angular velocity:
a – to definition of the angular replacement;
b – to definition of the vector angular velocity.

it is called by the *angular velocity vector*. From introduced relation it is possible to define a direction of the angular velocity vector (see figure 3.7b).

Namely because the rotation angle lies at the rotation axis as one can see from the relation mention above the angular velocity vector lies at the same axis too. As the rotation axis it is convenient to choose axis «z» so we have

$$\Delta\vec{\varphi}\left(\Delta\varphi_x = 0, \Delta\varphi_y = 0, \Delta\varphi_z = \Delta\varphi\right).$$

Hence we may conclude that the direction of the angular velocity vector coincides with the positive axis «z» direction if the rotation angle over time increases. If this angle decreases over time the direction of the angular velocity vector is opposite to the positive axis «z» direction. So in our case one may write

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\varphi_z}{\Delta t} = \frac{d\varphi_z}{dt}$$

(rest of projections of the angular velocity vector are equal to zero). Similarly, we may introduce a quantity

$$\vec{\varepsilon} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt}, \qquad (3.38)$$

which is called by the *angular acceleration*. In the case under consideration this vector has projections $\vec{\varepsilon}\left(\varepsilon_x = 0, \varepsilon_y = 0, \varepsilon_z \neq 0\right)$. The relation $\varepsilon_z = \frac{d\omega_z}{dt} > 0$ takes place if projection ω_z increases over of time and the inequality $\varepsilon_z = \frac{d\omega_z}{dt} < 0$ fulfils in the otherwise.

3.2.6.3 Connection between the linear and angular quantities. From the beginning we will find the connection between the linear and angular displacements. With this purpose we will turn to the figure 3.8a. In this figure r is the circle radius along which the particle rotates $\Delta\varphi$ is its rotation angle over space of time Δt. This angle coincides with the central angle of the circle mentioned above therefore we can write $\Delta S = r \cdot \Delta\varphi$ where ΔS is a length of the circle arc upon which given angle based on. In given figure O is the coordinates origin, C is centre of circle, $\vec{R}(t)$ is the particle radius-vector in the point of time t, $\vec{R}(t+\Delta t)$ is the consequence quantity in the point of time $t+\Delta t$, $\Delta\vec{R}$ is a linear displacement vector, α is the angle formed by the rotation axis (axis «z») and by the particle radius-vector $\vec{R}(t)$ (or $\vec{R}(t+\Delta t)$).

In consequence of smallness of ΔS and of $\left|\Delta\vec{R}\right|$ we may accept $\Delta S \approx \left|\Delta\vec{R}\right|$ and then we will write

$$\left|\Delta\vec{R}\right| = r \cdot \Delta\varphi. \tag{3.39}$$

Further from the figure 3.8a we see that $r = R \cdot \sin\alpha$ and therefore for equality (3.39) we have

$$\left|\Delta\vec{R}\right| = \Delta\varphi \cdot R \cdot \sin\alpha. \tag{3.40}$$

It is easy to see that right hand of the equality (3.40) represents the vector product of the vectors $\Delta\vec{\varphi}$ and $\vec{R}$. By other words

$$\Delta\vec{R} = \Delta\vec{\varphi} \times \vec{R}. \tag{3.41}$$

Using by the figure 3.8a and by known geometrical theorem about three perpendiculars one can make sure that the order of the multipliers in the relation (3.41) is chosen correctly. This equality represents the required relation between the linear and angular displacements.

From the relation (3.41) we will obtain the connection between the vectors of linear and angular velocities. Namely using the equalities (3.37), (3.39) we will have

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{R}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\varphi}}{\Delta t} \times \vec{R} = \vec{\omega} \times \vec{R}. \tag{3.42}$$

From figure 3.8b we determine the direction of the vector $\vec{v}$.

Obtained relation represents the required relation between the linear and angular velocities vectors. It is possible to use this relation for finding connection between the modulus of the linear and angular velocities vectors. Indeed from the relation (3.42) it is possible to write

$$v = \lim_{\Delta t \to 0} \frac{\left|\Delta\vec{R}\right|}{\Delta t} = r \cdot \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \omega r. \tag{3.43}$$

It is the required connection between the vectors of linear and angular velocities.

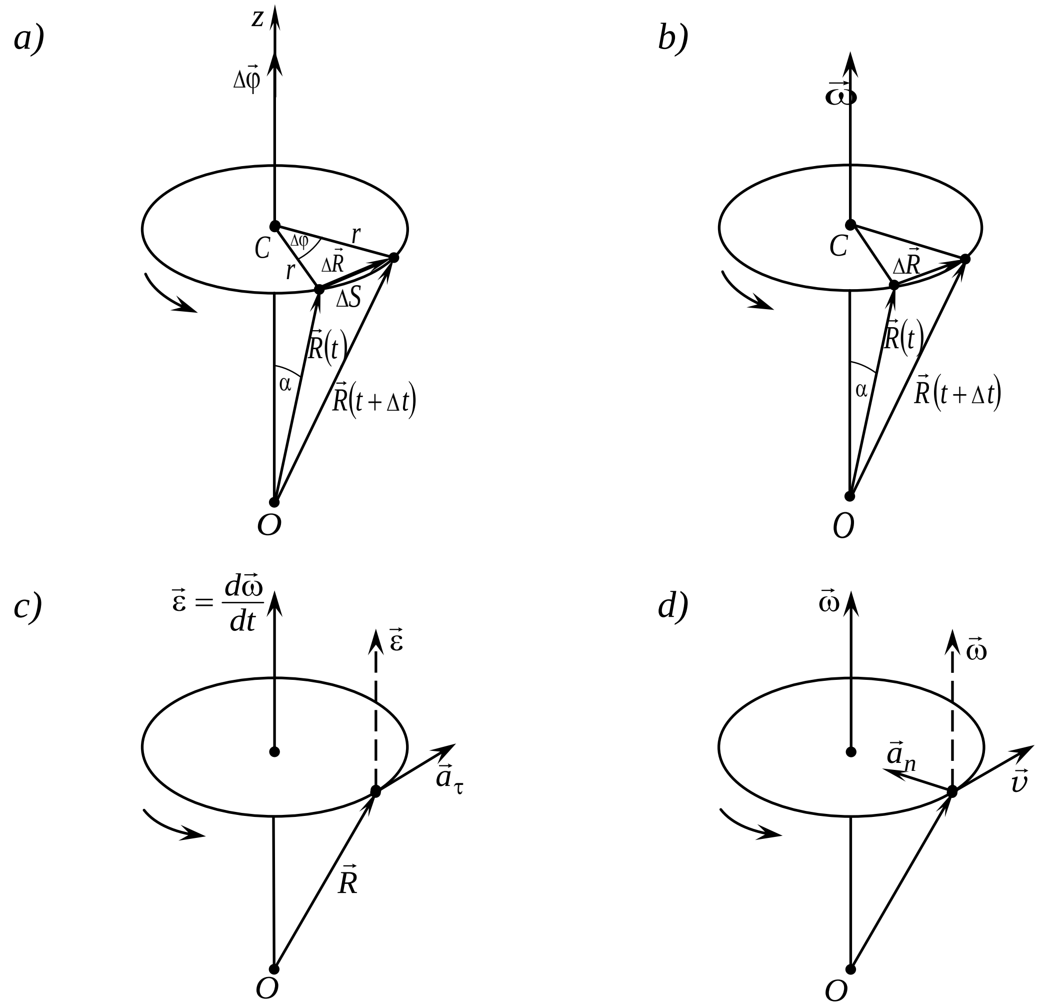

Figure 3.8. Establishing of the connection between the linear and angular values:
a – connection between the linear and angular displacement vectors;
b – connection between the linear and angular velocities;
c – connection between the linear and angular accelerations;
d – connection between the linear velocity, angular velocity and normal acceleration vectors.

Now we enter to establishing the connection between the vectors of linear and angular accelerations. With this purpose we will initiate from the expression (3.14), using the relation (3.42) and the rules of the product differentiation. Then we obtain

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{R} + \vec{\omega} \times \frac{d\vec{R}}{dt}.$$

From relations (3.38) and (3.42) it is possible to write obtained result in the form

$$\vec{a} = \vec{\varepsilon} \times \vec{R} + \vec{\omega} \times \vec{v}.$$

Using the relation (3.19), (3.38) we see that

$$\vec{a}_\tau = \frac{d\vec{\omega}}{dt} \times \vec{R}$$

is the tangential component of the linear acceleration, and

$$\vec{a}_n = \vec{\omega} \times \vec{v}$$

is its normal component. For modulus of these quantities we have

$$a_\tau = \left|\vec{\varepsilon} \times \vec{R}\right| = \varepsilon \cdot R \cdot \sin\alpha = \varepsilon \cdot r\,, \qquad (3.44)$$

$$a_n = \omega \cdot v = \omega^2 \cdot r = \frac{v^2}{r}.$$

Last formula coincides with formula (3.36) obtained early (**3.2.5.4)**. With the help of the figures 3.8c and 3.8d it is possible to be convinced of the choice of the directions of the vectors $\vec{a}_\tau$ and $\vec{a}_n$ is in the agreement with one which was defined in (**3.2.5.4**).

3.2.6.4 Remarks about rigid body kinematics which has fixed rotational axis

First of all we will introduce the concept of the *rigid body*. *By rigid body one means the particles system in which the distances between any couple of them are assumed to be constant.* It is obviously that the concept introduced by such a manner is a physical model in which we neglect by the possibility of the body deformation. There are rather large number of problems for which noticed neglecting is quite acceptable. One from these problems is the kinematics of the rigid body, and in particularly in the case when the last has the fixed axis of rotation.

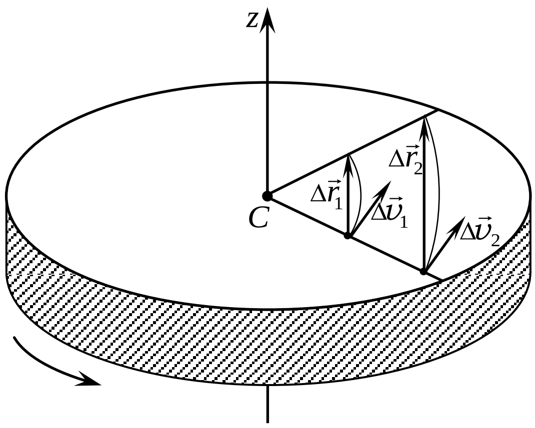

Figure 3.9. Linear quantities of the different points of the rigid body

Considering kinematics of the rigid body motion which has a fixed rotational axis we would note that in this case such linear characteristics as radius-vector, displacement, linear velocity and acceleration are meaningless for the total body. As one may convince of the figure 3.9. linear characteristics mentioned above for different body points are different too. At the same time there are kinematical quantities which may be chosen as characteristics of the total body state and of its motion. *Such characteristics are the angular quantities*. Namely as kinematical characteristics of the state and of the motion rigid body having fixed axis of rotation we take the angular displacement, angular velocity and angular acceleration, which were introduced for the material point in **3.3.1.1** and **3.3.1.2**.

3.2.7 Number of freedom degrees of the mechanical system. Concept on the generalized coordinates.

3.2.7.1. Concept of number of freedom degrees of the mechanical system.

In paragraph **3.2.1** it was noticed that coordinates of a particle are the quantities which enter in the number of its state characteristics. So a mechanical state of a free particle moving in three-dimensional space is characterized by three coordinates. However, there may be cases when not all of these coordinates are independent on. So, when a particle moves along the surface of its coordinates two of them are independent on only: if particle coordinates x and y are specified its coordinate z can be determined from the surface equation

$$z = f(x, y).$$

The same situation takes place when the particle moves along the line defined by the system of equations

$$\begin{cases} f_1(x, y, z) = 0; \\ f_2(x, y, z) = 0. \end{cases} \tag{3.45}$$

If one solves one of given equations e.g. the second one, relative the variable z he will obtain

$$z = g(x, y).$$

By inserting of obtained expression into the first of equations (3.45) we have

$$f_1(x, y, g(x, y)) = 0,$$

whence we see that it is possible to determine one from variables x and y if another variable is given. These reasoning show that if the particle moves along the arbitrary curve of its coordinates describing its state in the space is independent on only one of them:

Given examples show that in some cases not all particle coordinates it is necessary to define when its mechanical state is specified. This allows entering a new concept for describing of the position of the particle in space. In other words we will enter concept of the *number of freedom degrees of the particle meaning under this concept a number of independent on variables required for total describing of the particle position in space.* From given examples we see that free particle in three-dimensional space has three degrees of freedom, the particle on the surface has two degrees of freedom, and the particle motion of which is restricted by some curve has one degree of freedom only.

It is not difficult to generalize given reasoning on an arbitrary system of the particles. From the beginning we will consider the system consisting of two particles. In general this system has six degrees of freedom that are the coordinates of the particles However, if the particles are coupled by the rigid coupling a number of the system freedom degrees decreases. Really, in this case, as it is known from geometry, a distance between the particles with coordinates x_1, y_1, z_1 and x_2, y_2, z_2 (figure 3.10a) is determined by the expression

$$l_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2, \tag{3.46}$$

from which it follows that one from six of particles coordinates may be determined if rest of five from them are known. So, we see that in the case under consideration five from six coordinates only may be considered as independent on. Then we may say that the system under consideration has five freedom degrees.

Let us consider the system consisting of three particles. It is easy to see that in general case this system position in space is characterized by nine quantities $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$ (figure 3.10b). which are the coordinates of corresponding particles. If a rigid couple is imposed on the particle, in addition to the relationship (3,46) we may write

$$l_{23}^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2;$$
$$l_{13}^2 = (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2. \qquad (3.47)$$

Given relations and relation (3.46) allow us to determine three coordinates of the particles if rest of them are known (instead of one from given relationships one may specify the angle ϑ between any of lines shown in the figure 3.10b). Therefore one may say that given system has six degrees of freedom.

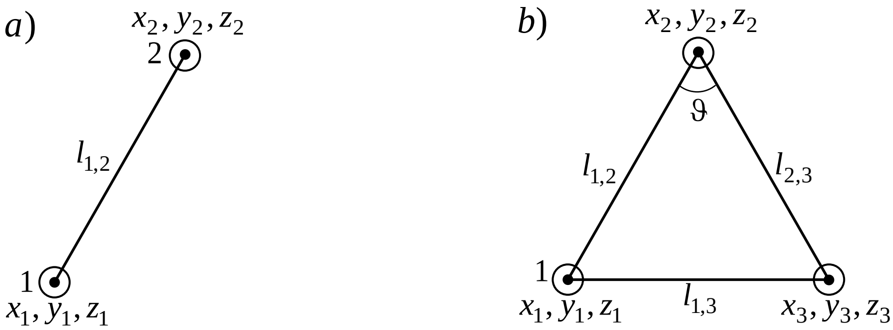

Figure 3.10. To calculation of the freedom degrees number of the particles system:
a – system consisting of two particles;
b– system consisting of three particles.

Summarizing, we say that the mechanical system consisting of N particles, has $f = 3N - r$ degrees of freedom, where r is the number of additional relations imposed on the coordinates of the particles. These relationships are usually called as *couplings*.

3.2.7.2. Spherical and cylindrical coordinates and their connection with Cartesian ones.

Although mentioned above Cartesian coordinates other ones are used. As the example of these coordinate systems the spherical and cylindrical coordinates will be considered. From the beginning we will return to the spherical coordinates. In this case a particle position in space is specified by its distance r from some point called as centre, by polar angle ϑ and azimutal angle φ (see figure 3.11a). The connection between the rectangular coordinates and spherical ones is given by the following relations:

$$x = r \sin\vartheta \cos\varphi;$$
$$y = r \sin\vartheta \sin\varphi;$$
$$z = r \cos\vartheta.$$

In the cylindrical coordinate system a particle position in space is specified by its distance r from some axis (axis *oZ*), by the azimutal angle φ, and by the coordinate z, modulus of which is equal to the particle distance from the plane $z = 0$ The connection between the rectangular and cylindrical coordinates is given by the following relations:

$$x = r\cos\varphi;$$
$$y = r\sin\varphi;$$
$$z = z.$$

This connection is shown on the figure 3.11b.

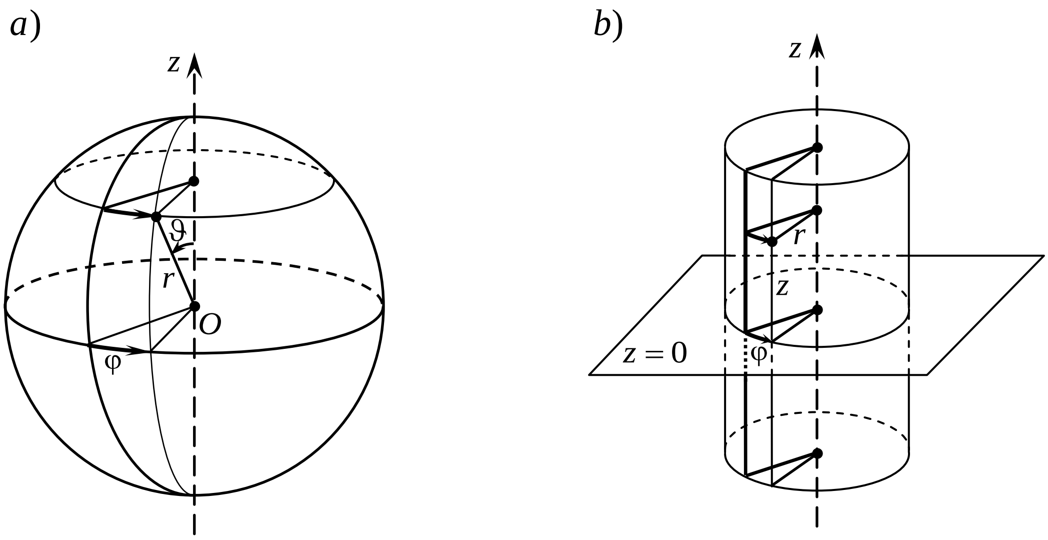

Figure 3.11. Coordinates different from Cartesion ones:
a – spherical coordinates;
b – cylindrical coordinates.

3.2.7.3. Number of freedom degrees of rigid body. For solving of great number of physical problems it is necessary to know the number of freedom degrees of rigid body. In this connection we will make an attempt to calculate mentioned quantity below. We will to origin from the assertion that a rigid body is such particles system in which the position of ever particle is specified by three coordinates. Therefore if system consists of *N* particles the position of total body may be specified by 3*N* coordinates of particles forming body which is under consideration. However, not all coordinates from 3*N* mentioned above are independent on. Really, initiating from the rigid body definition, for coordinates of any couple of its particles with numbers «*i*» and «*k*» a relation exists, which is similar to (3.46) and (3.47):

$$l_{ik}^2 = (x_i - x_k)^2 + (y_i - y_k)^2 + (z_i - z_k)^2. \tag{3.48}$$

As it follows from written relation it allows defining one from particle coordinates of number «*i*» if other quantities in given relation are known. Then we see that in order to definition all particle coordinates one has to consider three of all possible relationships only which are similar to (3.48) and are relevant to considered particles couple.

For solving formulated problem we will give all sorts of particles pairs couplings in the system which is under consideration:

$$
\begin{array}{cccccc}
l_{1,2}, & l_{1,3}, & l_{1,4}, & \ldots & l_{1,N-1}, & l_{1,N}; \\
 & l_{2,3}, & l_{2,4}, & l_{2,5}, & l_{2,N-1}, & l_{2,N}; \\
 & & & l_{N-3,N-2}, & l_{N-3,N-1}, & l_{N-3,N}; \\
 & & & & l_{N-2,N-1}, & l_{N-2,N}; \\
 & & & & & l_{N-1,N}.
\end{array}
$$

This table contains $N-1$ rows. From given table one can see that for determination of the coordinates x of $N-1$ particles it is enough to use the first elements of every row if the rest of coordinates are specified. This gives $N-1$ relationships. By the same operation using the second elements of every row we may obtain $N-2$ relationships for determination of the coordinates y of $N-2$ particles. Finally, for obtaining particles coordinates z we will use the third elements of every row of our table if the rest of coordinates are assumed to be specified, as it was assumed in the previous cases and as a result of this operation we will obtain $N-3$ relationships for $N-3$ particles. Then we have $r = N-1+N-2+N-3 = 3N-6$ of relationships which are the couplings between the coordinates of rigid body particles and in according to definition of the number of freedom degrees of the mechanical system we may say that free rigid body has

$$f = 3N - r = 3N - (3N-6) = 6$$

degrees of freedom. So, free rigid body has six degrees of freedom. It is interest to note that a system consisting of three particles which do not lie on the same straight line has the same number freedom degrees (see the equality (3.47) and reasoning after it). From this assertion it is possible to conclude that *position of any free rigid body in space may be specified by position of its any three particles not lying on the same straight line.*

From other hand we may say that *position of free rigid body in space is characterized by specification of three coordinates of its centre of inertia and three rotational angle of mutual perpendicular axes connected with body, which form these axes with the coordinate ones.*

Rigid body rotating around a fixed point has three of freedom degrees, and body that rotates around immobile axis, has one of the freedom degree only. This degree of freedom is the rotating angle.

3.2.7.4. The concept of the generalized coordinates

For solving of great quantity of physical problems it is convenient to describe the mechanical system by such number of coordinates which corresponds to the number of its degrees of freedom. As an example we will consider the system which consists of two particles which are connected one with another by the rigid coupling and which as it was noticed early (text after formula (3.46)) has five degrees of freedom. The position of the first of the particle we will specify by its three Cartesian coordinates. Connecting the origin of the spherical coordinates system with the first of the particle we will specify the position of the second of particle by the angles ϑ and angle φ which were entered earlier (see figure 3.11a).

Independent parameters which describe the position of the mechanical system in space and which number are equal to number of their degrees of freedom are called as generalized coordinates. Concept of generalized coordinates is widely used in quantum mechanics, statistical physics and thermodynamics and other chapters of physical science. In particular this concept is widely applied in solid physics when the mechanical, electrical, magnetic and thermal properties of solid are described.