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ABOUT THE CHIRAL SYMMETRY BREAKING IN QED_3

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The problem of the chiral symmetry breaking in QED_3 is considered by solving the Schwinger–Dyson equation for the fermion propagator in the ladder approximation using the Landau gauge for the photon propagator. Within the framework of the indicated approximation, different simplifications that allow expressions for the fermion mass function to be retrieved in an explicit form are analyzed. The results obtained are compared with the data of numerical analysis. It appears that the neglect of higher Gegenbauer harmonics in the kernel of the initial integral equation for the fermion mass function influences the dynamic mass value and the asymptotics of the mass function only weakly. On the other hand, it is established that the conclusion about a complicated structure of the fermion vacuum of the massive phase is an artifact of linearization of the Schwinger–Dyson equation kernel: consideration of the kernel nonlinearity yields a simple massive phase structure of the fermion vacuum.

Keywords: chiral symmetry, Schwinger–Dyson equation, ladder approximation, mass function, dynamic fermion mass, fermion vacuum.

1. Quantum electrodynamics in three-dimensional space (QED_3) has long drawn attention of many researchers. First, it is one of the few renormalizable theories. Second, the chiral symmetry braking takes place here [1–3] together with the spatial and temporal parities [4]. Third, confinement is observed in QED_3 for some approximations [5–8], which allows this phenomenon to be studied in more details compared to the QCD because of a simpler model. There are also grounds to believe that progress will be made in understanding of some macroscopic effects, in particular, high-temperature superconductivity. QED_3 is used to a certain degree to explain the structure of the elementary fermion excitation spectrum of the recently discovered layered structure – graphene [9, 10].

In the present work, the problem of the chiral symmetry breaking in QED_3 is considered. Comparing from this point of view QED_3 and QED_4, we note that there are no dimensional parameters in the latter that can be used to express the particle mass. Therefore, the dynamic mass arising here due to the chiral symmetry breaking is proportional to the cutoff parameter [11–14]. In QED_3, there is such parameter. On the other hand, in the context of the perturbation theory, the integral of the fermion energy in QED_3 diverges logarithmically, which suggests that the dynamic mass can be expressed through the cutoff parameter as well. Thus, the scale competition takes place in QED_3, and the question arises about momentum values that play the main role in the chiral symmetry breaking.

The 4-component QED_3 representation is considered below. The Schwinger–Dyson equations that must be solved by a non-perturbative method are used to analyze the symmetry breaking. Thus, the chiral symmetry breaking in QED_4 is observed already in the simplest non-perturbative ladder approximation if the coupling constant exceeds a critical value [11–14]. In this case, there exists a close analogy between the phenomena of incidence in the center in a strong Coulomb field for the one-particle problem of relativistic quantum mechanics and the chiral symmetry breaking in QED_4 [11]. As to QED_3, it has no spatial region in which the kinetic and potential components of the particle Hamiltonian in a static field would compensate each other disregarding the polarization of vacuum. In the case of massless dynamic fermions, the polarization of vacuum results in shielding of the static field at large distances and

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changing of the character of its dependence on the distance to the source; in this case, the region arises in which the static potential $A_0(r) \sim 1/r$, and the above-indicated compensation is observed [1] (in this case, the critical parameter of the model is the number of flavors N).

Such naive approach to the problem is insufficient, and the ladder approximation for the fermion propagator in QED_3 should be studied in more details. In particular, the question arises about simplifications of the ladder approximation that remain essential beyond its limits. The question on the structure of the fermion vacuum in QED_3 in the ladder approximation is also closely related with it. Finally, of interest is to find an analytical solution that will retain valid all results obtained by the numerical method including, in particular, the structure of the fermion vacuum in QED_3.

In the present work, the Schwinger–Dyson equation is solved for the fermion propagator and the dynamic fermion mass is determined with the use of the ladder approximation and simplifications described below (see items 2*a*, 2*b*, and 2*c*).

2. We start from the Schwinger–Dyson equation for the fermion mass function in the ladder approximation with the Landau gauge for the photon propagator. Then we have [2]

$$M(p) = \frac{2e_0^2}{(2\pi)^3} \int \frac{M(k)}{k^2 + M^2(k)} \frac{d^3k}{l^2} . \tag{1}$$

Here e_0^2 is the dimensional coupling constant in QED_3, $l = p - k$, p and k are three-dimensional Euclidean vectors, and M is the mass function of interest to us. Going to the Euclidean variables and integrating Eq. (1) over angles, with the chosen gauge we obtain

$$M(p) = \frac{2e_0^2}{(2\pi)^2 p} \int_0^\infty \frac{M(k)}{k^2 + M^2(k)} \ln\left|\frac{p+k}{p-k}\right| kdk . \tag{2}$$

This equation was considered by many authors. In particular, in [2] it was solved by a numerical method. The results obtained there were presented in the form inconvenient for an analysis. In addition, it is desirable to find explicit forms of the mass function and dynamic mass that are lacking from [2]. Therefore, we further use some simplifications of Eq. (2) enumerated below.

a) In sums over n in the expansion

$$\ln\left|\frac{p+k}{p-k}\right| = 2\theta(p-k) \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{k}{p}\right)^{2n+1} + 2\theta(k-p) \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{p}{k}\right)^{2n+1} , \tag{3}$$

only the first terms are taken into account (this is equivalent to the consideration only of harmonics with $n = 0$ in the expansion of $1/l^2$ in the Gegenbauer polynomials). In addition, considering that the inequality $M^2(k) \ll k^2$ is satisfied for large k^2, we eliminate the term with $M^2(k)$ from the denominator of the integrand of Eq. (2). As a result, replacing the lower integration limit in this equation by the dynamic mass m, we obtain for the mass function

$$M(p) = \frac{e_0^2}{\pi^2} \left(\frac{1}{p^2} \int_m^p M(k)\,dk + \int_p^\infty \frac{M(k)}{k^2} dk \right). \tag{4}$$

Equation (4) has both zero and nonzero solutions for $M(p)$. To find nonzero solutions using the normalization condition

$$M(p = m) = m , \tag{5}$$

we represent Eq. (4) in the form

$$M(p) = m + \frac{e_0^2}{\pi^2}\int_m^p \left(\frac{1}{p^2} - \frac{1}{k^2}\right) M(k)\,dk\,, \tag{6}$$

where

$$m = \frac{e_0^2}{\pi^2}\int_m^\infty \frac{M(k)}{k^2}\,dk\,. \tag{7}$$

This equality is the self-consistency condition from which the fermion mass can be determined.

Equation (6) can be solved in the context of the perturbation theory with the m value chosen as a zero approximation, whereas the zero approximation of the perturbation theory for Eq. (4) is equal to zero. This circumstance is decisive for the symmetry breaking in the examined case, since the choice of Eq. (5) for the boundary condition already suggests the chiral symmetry breaking.

Relation (6) is equivalent to the differential equation

$$p^2\frac{d^2M}{dp^2} + 3p\frac{dM}{dp} + \frac{2e_0^2}{\pi^2}p^{-1}M(p) = 0 \tag{8}$$

with boundary condition (5) as well as condition

$$\left.\frac{dM}{dp}\right|_{p=m} = 0\,. \tag{9}$$

The general solution of this equation has the form

$$M(p) = p^{-1}\left(c_1 J_2\left(\alpha_p\right) + c_2 N_2\left(\alpha_p\right)\right), \tag{10}$$

where J_2 and N_2 are the Bessel and Neumann functions, respectively, and $\alpha_p = \left(8e_0^2/\pi^2 p\right)^{1/2}$. Using Eqs. (5) and (9), we obtain

$$M(p) = \frac{\pi m^2}{2p}\alpha_m\left(N_1(\alpha_m)J_2\left(\alpha_p\right) - J_1(\alpha_m)N_2\left(\alpha_p\right)\right), \tag{11}$$

where $\alpha_m = \left(\frac{8e_0^2}{\pi^2 m}\right)^{1/2}$. Substitution of Eq. (11) into Eq. (7) yields the relation

$$J_1(\alpha_m) = 0\,, \tag{12}$$

that is, $\alpha_m = j_{1,n}$ are roots of the function $J_1(x)$ of order n (except zero). It is well known that this function has infinite number of simple roots; moreover, all of them are real and positive. Therefore,

$$m = m_n = \frac{8e_0^2}{\pi^2 j_{1,n}^2}\,. \tag{13}$$

From Eq. (12) it follows that $c_2 = 0$, and the fermion mass function in the considered approximation is

$$M(p) = \frac{\pi m_n^2 \alpha_m}{2p} N_1(\alpha_m) J_2(\alpha_p) = \frac{m_n^2}{p} \frac{J_2(\alpha_p)}{J_2(\alpha_m)}. \tag{14}$$

Thus, we obtained the solution presented in [2] to within a constant factor. This factor here was determined by solving the Cauchy problem for Eq. (8) with conditions (5) and (9) and self-consistency condition (7), that is, it is unambiguously related to Eq. (12). We note also that for α_m values at which the function $J_1(\alpha_m)$ is equal to zero, the function $N_1(\alpha_m) \neq 0$ and hence $M(p) \neq 0$, that is, the chiral symmetry has been broken. Equality (14) also demonstrates that $M(p) \sim 1/p^2$ when $p \to \infty$, which confirms the correctness of the neglect of $M^2(k)$ in formula (2).

From Eqs. (13) and (14) we can conclude that the chiral symmetry breaking takes place in this model and that the fermion vacuum in it has a complicated structure. Due to the principal importance of this point, we now consider some possible improvements of the approximation used here to solve Eq. (1).

b) After replacement $M^2(k) \to m^2$ in the denominator of Eq. (1) and introduction of the electronic propagator component $G_M(k) = \dfrac{M(k)}{k^2 + m^2}$ that does not contain the Dirac matrices, we obtain for this component the relation

$$\left(p^2 + m^2\right) G_M(p) = \frac{2e_0^2}{(2\pi)^3} \int \frac{G_M(k)}{l^2} d^3k. \tag{15}$$

This formula is a direct analog of the integral equation for the wave function of the hydrogen atom in the *p*-representation studied for the first time by V. A. Fok (see [15]). He demonstrated that solutions (15) are four-dimensional spherical functions. We will not present here all solutions of this equation and consider only *s*-symmetric solutions of interest to us that have the form

$$G_M(p) = C\left(p^2 + m^2\right)^{-2} \frac{\sin n\gamma}{\sin\gamma}, \tag{16}$$

where $\gamma = \arccos\dfrac{m^2 - p^2}{m^2 + p^2}$, $n \in N$, and C is a constant determined by condition (5). Then we have

$$M(p) = \frac{m^2}{\sin(n\pi/2)} \frac{1}{p} \sin\left(n \arccos\frac{m^2 - p^2}{m^2 + p^2}\right). \tag{17}$$

It is obvious that the expression presented above makes sense only for odd *n*. On the other hand, from the properties of four-dimensional spherical functions it follows that a nonzero solution to Eq. (15) exists only when the condition

$$m = \frac{e_0^2}{4\pi n} \tag{18}$$

is satisfied, that is, as a matter of fact, equality (17) describes a family of solutions, and Eq. (18) yields a spectrum of masses.

We note that Eq. (15) has nonzero solutions for $m^2 < 0$. These solutions describe unstable massless states, that is, they testify to the existence in this case of the massive phase described by Eq. (18). To answer the questions on the

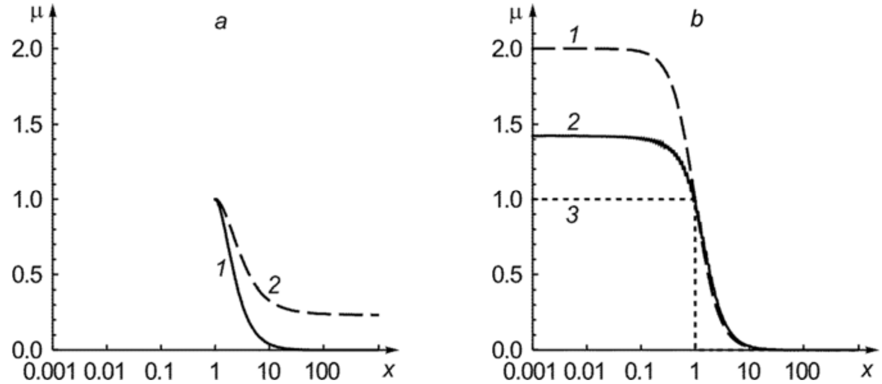

Fig. 1. Plots of dependences of the mass function $\mu = M/m$ on $x = p/m$ in the logarithmic scale corresponding to *a*) Eq. (14) (curve *1*) and numerical solution of Eq. (23) (curve *2*) and *b*) Eqs. (17) (curve *1*) and (19) (curve *3*) and numerical solution of Eq. (2) (curve *2*).

degree to which this circumstance reflects the presence of the fermion vacuum having a complicated structure in QED_3 and whether the results given by Eqs. (17) and (18) are artifacts of the employed approximation, it is desirable to consider a solution of Eq. (1) with allowance for the nonlinearity. This problem is considered below.

c) The problem can be solved somewhat differently if we seek a solution to Eq. (1) in the form

$$M(p) = M(0)\theta(m-p) \tag{19}$$

with the normalization condition

$$M(p=m) = m = M(0). \tag{20}$$

For nonzero m, this yields

$$1 = \frac{2e_0^2}{(2\pi)^3}\int_0^m \frac{d^3k}{(k^2+m^2)k^2}. \tag{21}$$

After integration over angles, for the dynamic mass we obtain

$$m = \frac{e_0^2}{4\pi} = 0.0796e_0^2. \tag{22}$$

Expression (19) is the unique solution of Eq. (1); in this case, Eq. (22) coincides with the mass value obtained in item b) for $n = 1$. The uniqueness of solution (19) for nonzero mass in the examined approximation prompts that a set of solutions for the dynamic mass and hence for the mass function in the approximations considered above can be the artifact of these approximations.

3. Let us supplement the analytical consideration of the problem with numerical one whose results are illustrated by Fig. 1. In particular, of interest is a solution of the Cauchy problem with conditions (5) and (9) for the nonlinear equation

$$p\left(p^2+M^2(p)\right)\frac{d^2M}{dp^2}+3\left(p^2+M^2(p)\right)\frac{dM}{dp}+\frac{2e_0^2}{\pi^2}M(p)=0 \tag{23}$$

derived from Eq. (2) if we ignore all terms in sums (3) except the first ones, but consider the term $M^2(k)$ in the integrand of the denominator of Eq. (2). To solve the equation, we go to the dimensionless variable $x = p/m$ and dimensionless function $\mu(x) = M(p)/m$. In this case, for the dynamic mass we have

$$m = 0.0734 e_0^2 . \tag{24}$$

The mass function corresponding to this mass is shown in Fig. 1*a*. To simplify an analysis of Fig. 1, the plots of the dimensionless mass functions obtained by numerical integration of Eq. (2) and the functions corresponding to solutions of Eqs. (17) and (14) (for $n = 1$) as well as Eqs. (19) and (23) are shown in the figure.

Let us consider now the numerical solution of integral equation (2). In dimensionless variables, Eq. (2) assumes the form

$$\mu(x) = \frac{\lambda}{2x} \int_0^\infty \frac{\mu(y)}{y^2 + \mu^2(y)} \ln\left|\frac{x+y}{x-y}\right| y dy , \tag{25}$$

where $\lambda = e_0^2 / (\pi^2 m)$. Condition (5) for the dynamic mass yields

$$m = 0.0710 e_0^2 . \tag{26}$$

The results of numerical investigation of the integral equation are shown in Fig. 1.

4. We now summarize the results of this work and briefly discuss the results obtained.

1) The nonzero solution of the Schwinger–Dyson equation for the fermion propagator has been obtained in this work in the ladder approximation with the Landau gauge that breaks the chiral symmetry of the initial Lagrangian. Within the limits of the ladder approximation, the linearized version of Eq. (1) was considered with allowance for both the first nonzero term in the expansion of the kernel $1/l^2$ in the Gegenbauer polynomials (item 2a) and all terms in this expansion (item 2b).

2) The approximate solution of nonlinear equation (1) was found (item 2c).

3) The numerical solution of Eq. (2) and nonlinear equation (23) with boundary conditions (5) and (9) (item 3) was found. In both cases, close values were obtained for the dynamic mass.

4) Expressions for the fermion dynamic mass were obtained in the ladder approximation using the Landau gauge for the photon propagator (formulas (13), (18), (22), (24), and (26)).

Analyzing the results obtained, we note the following.

– Unlike QED_4 [3, 11–14], there are no critical parameters indicating the presence of a phase transition for the examined ladder approximation, and the dynamic mass is observed at any arbitrary value of the coupling constant.

– The dynamic mass is formed mainly due to the interaction at distances $r \sim 1/m$. This circumstance differs radically from the 4-dimensional case (see [11–14]) in which distances $r \ll 1/m$ are responsible for the indicated process and from QED_3 with allowance for the polarization of vacuum of massless fermions, as pointed out in item1.

– At first glance, the family of solutions (14) and (17) for the mass function and the set of masses (13) and (18) testify to a complicated structure of the fermion vacuum of the massive phase. Since the mass value characterizes the vacuum reorganization time and the occurrence of the massive phase, there are grounds to believe that the presence of states with $n > 1$ is an artifact of the considered approximation. The absence of solutions corresponding to such states in numerical investigation of the problem (Eq. (23)) as well as in the simplest account for the nonlinearity in the kernel of integral equation (1) (see item 2c and item 3) testifies to plausibility of this statement. The absence of a complicated structure of the fermion vacuum of the massive phase is also confirmed by the numerical solution of integral equation (2).

– According to the foregoing, only masses with $n = 1$ must be taken into account in further analysis with the use of Eqs. (13) and (18). In this case, from Eq. (13) we obtain $m = 0.0552 e_0^2$. This value differs by 31% from the mass specified by Eq. (18) derived by linearization of Eq. (1) with allowance for all terms in expansion (3). It differs by 25% from the mass specified by Eq. (24) derived by neglecting all terms in expansion (3) except the first term, but with allowance for the nonlinear character of the initial equation. In this case, the difference from the value determined by Eq. (26) derived by numerical solution of Eq. (2) is 22%. Thus, the consideration of all terms in expansion (3) on the one hand and of the nonlinearity of initial equations (1) and (2) on the other hand give almost identical values of the dynamic fermion mass. Moreover, the letter results in the simple vacuum of the massive phase. This is preferable first, from the physical viewpoint and second, coincides with the results of numerical calculation.

One of the main problems arising in the study of the problem of the chiral symmetry breaking in QED_3, by analogy with QED_4, is a dependence of the results obtained on the gauge [2, 3]. Construction of approximations whose consequences are independent of the gauge is, in our opinion, one of the most important problems in this direction of research in quantum field theory.

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