LOCAL PROPERTIES OF NONRENORMALIZABLE INTERACTIONS

M. Sh. Pevzner

UDC 539.126

The possibility of constructing strictly localizable fields is considered, using generalized S-type spaces as spaces of basis functions. The restrictions imposed on the asymptotic behavior of the amplitudes coincide with the well-known restrictions found by Jaffe. Contrary to previous results, the spatial amplitudes and momenta are regular, and not singular functions, in the case considered. The possibility of formulating spectral conditions is investigated.

1. Objects studied by the quantum theory of fields are various ordered products of field operators averaged over the vacuum. The latter are linear, continuous functions, given in some space of basis functions. This is a mathematical reflection of the fact that the observables are averages over some region of space–time of some field operator $A(x)$, i.e., of an operator of the form

$$A(\varphi)=\int A(x)\varphi(x)(dx). \tag{1}$$

The Schwartz space (S) or the space of infinitely differentiable functions with a compact support (K) are usually chosen for such a space. Studying nonnormalizable interactions, it was discovered that the Whiteman and Green functions may grow exponentially in momentum space [1-8] and, consequently, are not linear continuous functionals on S. In this connection there arises the problem of the space of basis functions to be used in the axiomatic quantum theory of fields in studying nonnormalizable interactions.

The first of these problems was stated by Meiman [9]. He determined the structure of the space of basis functions, starting from Bogolyubov's microcausality principle [10]. Using such an approach it is possible to eliminate the assumption of moderate growth of generalized functions in the quantum theory of fields, which is a purely mathematical assumption, having no relation to the physics of the problem, and to extend the results, obtained by axiomatic and dispersion methods, to the case when generalized functions are not moderately distributed (see also [11]).

A different way of solving the problem was suggested by Jaffe [12]. He pointed out that the vectors $A(\varphi)\Psi_0$ (Ψ_0 is the vector of the vacuum state) form a Hilbert space. Accounting for the spread, the locality or microcausality principle should be formulated as

$$[A(\varphi_1),\ A(\varphi_2)]_{\pm}=0, \tag{2}$$

if the regions where $\varphi_1(x)$ and $\varphi_2(x)$ do not vanish are spatially similar. For (2) to make sense it is necessary that the averages of the basis space functions be sufficiently finite functions. This space, however, cannot coincide with K since, in particular, it does not allow the condition of spectrality to be formulated. The Fourier transformations of the generalized functions studied by Jaffe assume an increase of the form

$$g(p)\sim\exp\{\|p\|/\ln^{1+\varepsilon}\|p\|\}; \tag{3}$$

and in Meiman's approach the corresponding amplitudes assume the estimate

$$g(p)<\exp\varepsilon\|p\|,\ \|p\|=(p^2+p_0^2)^{1/2},\ \varepsilon>0. \tag{4}$$

In what follows we adhere to Fainberg's terminology [13], calling a theory assuming growth of type (3) strictly localizable, and a theory assuming growth of type (4) localizable.

Dnepropetrovsk Mining Institute. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 8, pp. 76-81, August, 1971. Original article submitted June 1, 1970.

If (2) is adopted as a locality criterion, the localizable and strictly localizable theories necessarily satisfy this criterion. The theories mentioned can, on the other hand, also be nonlocal.

Our problem involves the S-type spaces and their generalizations [14], more thoroughly studied from the point of view of the possibility of using them as basis function spaces in the local quantum field theory. A growth of type (3) is also assumed; however, contrary to Jaffe's results in the given case the generalized functions in momentum space are regular, and not singular, functionals. The problem of formulating the spectral condition can also be solved.

2. We consider a generalized S-type space [14] The space S_{a_κ} is defined as the set of functions $\varphi(x)$ satisfying the inequality

$$|x^\kappa \varphi^{(q)}(x)| \leqslant C_q A^\kappa a_\kappa; \tag{5}$$

a_κ is an arbitrary series of numbers; and C_q and A are constants, depending on $\varphi(x)$. The numbers a_κ impose restrictions on the decrease of the basis functions at $|x| \to \infty$.

The space S^{b_q} contains all functions $\varphi(x)$ satisfying the inequality

$$|x^\kappa \varphi^{(q)}(x)| \leqslant CB^q b_q. \tag{5a}$$

Here b_q is also an arbitrary series of numbers; and C_q and B are also dependent on $\varphi(x)$. The numbers b_q restrict the growth of derivatives of the functions $\varphi(x)$ with an increase in their order. Imposing at the same time restrictions on the attenuation of the basis functions at $|x| \to \infty$ and on the growth of derivatives with an increase in their order, we obtain functions belonging to the space $S_{a_\kappa}^{b_q}$. These functions satisfy the inequality,

$$|x^\kappa \varphi^{(q)}(x)| \leqslant CA^\kappa B^q a_\kappa b_q. \tag{5b}$$

The space $S_{a_\kappa}^{b_q}$ is the intersection of the spaces S_{a_κ} and S^{b_q}, i.e.,

$$S_{a_\kappa}^{b_q} = S_{a_\kappa} \cap S^{b_q}. \tag{5c}$$

It is easily derived from (5a)-(5c) that when the condition $\lim_{q\to\infty} b_q/q^q = 0$ is satisfied, any function $\varphi(x) \in S_{a_\kappa}^{b_q}$ can be continued to the band $|y| < 1/Be$ of the complex $z = x + iy$ plane; the band width is specific to every function $\varphi(x)$.

Let us draw attention to the following. All infinitely differentiable functions, vanishing outside a given segment of the real axis, which is characteristic of each function $\varphi(x)$, appear in the K space. At the same time the definition of this space does not impose any restrictions on the growth of the derivatives of these functions with increasing order. It is easy to see, therefore, that by imposing these or other restrictions we obtain another space of basis functions, which is not a subspace of K.

The space of generalized functions can in this case be larger than K' and is, consequently, admissible for our purposes. Such a space is, in particular, $S_1^{b_q}$.

3. The space $S_{a_\kappa}^{b_q}$ is the union of topological spaces $S_{a_\kappa, A}^{b_q, B}$, i.e.,

$$S_{a_\kappa}^{b_q} = \bigcup_{A=1}^{\infty} \bigcup_{B=1}^{\infty} S_{a_\kappa, A}^{b_q, B}. \tag{6}$$

In the spaces $S_{1,A}^{b_q, B}$ the topology may be given by the denumerable set of norms

$$\|\varphi\|_m = \sup_q \frac{|\varphi^{(q)}(x)|}{\left(B + \frac{1}{m}\right)^q b_q}, \quad |x| \leqslant A \tag{7}$$

(m = 1, 2, . . .). With these norms $S_{1,A}^{b_q, B}$ is a complete, denumerably normalized, ideal, nuclear space.

All preceding and following arguments remain valid when passing to a space of several variables, using the representation

$$\varphi(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} \varphi_i(x_i). \tag{8}$$

4. Consider now the space $S_{1,A}^{b_q, B}$. A linear continuous functional in this space is of the form [15]

$$(f, \varphi) = \sum_{q=0}^{\infty} \int D^q \varphi(x)\, d\sigma_q(x),$$
$$D^q \varphi(x) = \frac{\partial^{q_1+q_2+\dots+q_n} \varphi(x_1, x_2, \dots x_n)}{\partial x_1^{q_1} \partial x_2^{q_2} \dots \partial x_n^{q_n}}. \tag{9}$$

The measures $\sigma_q(x)$ are concentrated in the regions R_n, all points of which satisfy the inequality $|x_i| \le A_i$. The norm of functional (9) equals

$$|(f, \varphi)| = \sum_{q=0}^{\infty} \left(B + \frac{1}{m}\right)^q b_q \int d\sigma_q(x). \tag{10}$$

The continuity condition of functional (9) is equivalent to the finiteness of its norm, i.e.,

$$\sum_{q=0}^{\infty} \left(B + \frac{1}{m}\right)^q b_q \int d\sigma_q(x) < \infty, \tag{11}$$

which is a restriction on the order of the measures $\sigma_q(x)$.

The nuclearity of the space under consideration allows to introduce a new system of norms

$$\|\varphi\|'_m = \sup_q \frac{\int |D^q \varphi| (dx)}{\left(B + \frac{1}{m}\right)^q b_q}, \tag{12}$$

which is equivalent to (7) (with the distinction that the multidimensional case is considered in (12)). With this system of norms we obtain instead of (10) and (11)

$$(f, \varphi) = \sum_{q=0}^{\infty} \int D^q \varphi(x) f_q(x)\, (dx), \tag{13}$$

$$|(f, \varphi)| = \sum_{q=0}^{\infty} \left(B + \frac{1}{m}\right)^q b_q \sup_x |f_q(x)| < \infty; \tag{14}$$

$f_q(x)$ is a sequence of functions bounded and measurable in the region $|x_i| \le A_i$.

5. We consider the nontriviality of $S_{1,A}^{b_q, B}$. To this end, we note that the inequality

$$|\varphi^{(q)}(x)| \leqslant C\left(B + \frac{1}{m}\right)^q b_q \cdot \begin{cases} 1 & \text{if} \quad |x| \leqslant A, \\ 0, & \text{if} \quad |x| > A \end{cases} \tag{15}$$

holds for all functions appearing in $S_{1,A}^{b_q, B}$. Therefore the problem of the nontriviality of $S_{1,A}^{b_q, B}$ is a classical problem of quasianalyticity [14]. Conditions should be imposed on the numbers b_q, such that there exist infinitely differentiable functions $\varphi(x) \not\equiv 0$, vanishing outside a finite segment and satisfying the inequality

$$|\varphi^{(q)}(x)| \leqslant C\left(B + \frac{1}{m}\right)^q b_q.$$

The answer to the question posed is provided by the Carlman–Ostrovski theorem. For the space $S_{1,A}^{b_q, B}$ to be nontrivial a necessary and sufficient condition is

$$\int_1^{\infty} \frac{\ln \Gamma(x)}{x^2}\, dx < \infty, \tag{16}$$

where $\Gamma(x) = \max_q x^q / b_q$ is the Ostrovski function. (Another nontriviality criterion, and also a nontriviality criterion of the $S_{a_K}^{b_q}$ space, is considered in [16].)

Thus, if only $b_q \to \infty$ sufficiently quickly for $q \to \infty$, the $S_{1,A}^{b_q, B}$ space is nontrivial, contains finite functions, and does not contain analytic functions.

6. We consider now the problem of Fourier transforms of basis functions of the $S_{a_K}^{b_q}$ space and also of linear continuous functionals on this space. The following theorem [17] holds. Let the conditions $\underline{\lim}\, b_q^{1/q} = \infty$, $a_K < \infty$ be satisfied. Then $\tilde{S}_{a_K}^{b_q} = S_{b_K}^{a_q}$.

According to the theorem we write down

$$|p^{\kappa}\psi^{(q)}(p)| \leqslant C'A^{q}B^{\kappa}b_{\kappa}a_{q}, \tag{17}$$

$$|\psi^{(q)}(p)| \leqslant C'A^{q}\Gamma^{-1}(p/B)\,a_{q} \tag{17a}$$

(ψ is the Fourier transform of the function $\varphi(x)$). In what follows we confine the discussion to the case $a_q = a_K = 1$ (this case corresponds exactly to our statement of the problem). It is easily seen that the function

$$\Gamma(s) = P(s)\exp\{s/\ln^{1+\varepsilon}s\} \tag{18}$$

($P(s)$ is a polynomial of finite degree in s, $\varepsilon > 0$) satisfies condition (16). It can be verified that this condition prohibits a faster growth than (18). Hence follows the restriction on the growth of $\psi^{(q)}(p)$

$$|\psi^{(q)}(p)| \leqslant C'A^{q}P^{-1}(p/B)\exp\{-p/B\ln^{1+\varepsilon}p\}. \tag{19}$$

For the Fourier transforms of generalized functions we have

$$(g,\psi) = \sum_{q=0}^{\infty}\int (ip)^{q}g_{q}(p)\,\psi(p)\,(dp), \tag{20}$$

where $g_q(p)$ is a series of integral functions of degree of growth not exceeding 1.

We consider the series

$$G(p) = \sum_{q=0}^{\infty}(ip)^{q}g_{q}(p) \tag{21}$$

and we show that it converges uniformly in the whole plane of the complex variable p. With this purpose in mind we notice that by (14) the series $\sum_{q=0}^{\infty}\left(B+\frac{1}{m}\right)^{q} b_q|g_q(p)|$ converges for all p. Hence it follows rapidly that at $q \to \infty$ the quantity $C_q(p) = b_q|g_q(p)|$ has the limit

$$C_q^{1/q}(p) \to \text{const}. \tag{22}$$

It follows from the nontriviality of $S_1^{b_q}$ that the inequality

$$\sum_{q=0}^{\infty}(ip)^{q}g_{q}(p) < \sum_{q=0}^{\infty}\frac{(ip)^{q}}{q!}C_{q}(p) \tag{23}$$

holds, and from (22) the uniform convergence of the series investigated follows immediately. Since (21) is a uniformly convergent series of integral functions, its sum is also an integral function.

It is thus seen that in the case considered the linear, continuous functionals are regular functionals of the integral function G(p) type, becoming infinite along the real axis not faster than $P(p)\exp\{p/a\ln^{1+\varepsilon}p\}$. It is thus seen that in passing to the 4-dimensional space the estimate (3) is valid.

7. We discuss the problem of formulating the spectral condition. In its usual formulation this condition states that the average of field operators $\langle 0|A(x_1)\ldots A(x_n)|0\rangle$ over the vacuum contains contributions from states of positive energy only. For this it is necessary that the Fourier transforms of the space of basis functions $\varphi(x)$ be finite functions. It is easily seen that among the functions $\psi(p)$ belonging to $S_{b_K}^1$ there are no finite functions. However, in this case any continuous function φ with bounded support $G_\varphi \subset R_n$ can be approximated as closely as desired by an integral function $\psi \in S_{b_K}^1$, significantly differing from zero in the region $G_\psi = G_\varphi + iR^n$, and, outside some open region $G_\psi^1 \supset G_\varphi$, smaller than any given $\varepsilon > 0$ [18]. This suffices completely for formulating the spectral conditions.

8. It has thus been shown that the use of generalized S-type spaces in quantum field theory allows the derivation of the same restrictions on the asymptotic behavior of amplitudes outside mass surfaces as in the theory of strictly localizable fields [12]. The amplitudes mentioned define regular functionals. The possibility of local interpretation of interactions the matrix elements of which satisfy the restriction (3) outside the mass surface is thus confirmed.

LITERATURE CITED

1. R. Arnowitt and S. Deser, Phys. Rev., 100, 349 (1955); L. Cooper, Phys. Rev., 100, 362 (1955).
2. W. Güttinger, Nuovo Cim., 10, 1 (1958); W. Güttinger and E. Pfaffelhuber, Nuovo Cim., 52A, 389 (1967).

3. T. Pradhan, Nucl. Phys., 43, 11 (1963); S. Okubo, Nuovo Cim., 19, 574 (1961).
4. B. Schroer, J. Math. Phys., 5, 1361 (1964); K. Bardacki and B. Schroer, J. Math. Phys., 7, 10, 16 (1966).
5. Klaiber, Nuovo Cim., 36, 165 (1965).
6. T. Hadjioannou, Nuovo Cim., A42, 241 (1966).
7. M. Sh. Pevzner, Ukr. Fiz. Zh., 12, 100 (1967).
8. M. K. Volkov and G. V. Efimov, Zh. Éksp. Teor. Fiz., 47, 1800 (1964); M. K. Volkov, Yad. Fiz. 7, 445 (1968).
9. N. N. Meiman, Zh. Éksp. Teor. Fiz., 47, 1966 (1964).
10. N. N. Bogolyubov and D. V. Shirkov, Introduction to Quantum Field Theory, Interscience (1959); N. N. Bogolyubov, B. V. Medvedev, and M. K. Polivanov, Theory of Dispersion Relations, Lawrence Radiation Laboratory (1961).
11. Yu. M. Lomsadze, in: High-Energy Physics and Elementary Particle Theory [in Russian], Naukova Dumka, Kiev (1967).
12. A. Jaffe, Phys. Rev., 158, 1454 (1967).
13. V. Ya. Fainberg, Preprint FIAN, No. 137 (1968).
14. I. M. Gel'Fand and G. E. Shilov, Generalized Functions, Vol. 2, Academic Press, New York (1964).
15. B. S. Mityagin, Trudy Moskv. Matem. Obshch., 9, 319 (1960).
16. S. Mandelbroit, Quasianalytic Classes of Functions [Russian translation], ONTI (1937).
17. S. Mandelbroit, Theory of Closures and Composition Theorems [Russian translation], IL (1962).
18. G. V. Efimov, Preprint ITF-68-52.