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ON THE CHARACTER OF CHIRAL SYMMETRY BREAKING AND FERMION VACUUM STRUCTURE IN QED_3

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Equation for the Bethe-Salpeter wave function of the Goldstone boson in QED_3 is considered in the ladder approximation with the use of the Landau gauge for the photon propagator. With the help of standard simplifications, the existence of nonzero solutions for this equation is demonstrated, which testifies to the production of the above-described boson in the process of chiral symmetry breaking. At the same time, it is demonstrated that only one of the entire set of solutions describing the Goldstone boson corresponds to the stable ground state; this solution has the greatest fermion mass. In the remaining cases, the compound boson state with zero mass is excited, and all other states having smaller energies appear tachyon states and hence are unstable. The fermion condensate is calculated; it is demonstrated that in the examined case, it is finite. Based on the foregoing, conclusions are drawn about spontaneous rather than dynamic character of chiral symmetry breaking in QED_3, complex structure of fermion vacuum for the examined model, and at the same time, simple structure of the massive phase vacuum.

Keywords: chiral symmetry, Bethe–Salpeter wave function, ladder approximation, Goldstone boson, tachyon, dynamic fermion mass, fermion condensate, fermion vacuum.

1. The problem of chiral symmetry breaking in QED_3 was studied in [1]. The main method of investigations there was a solution of the Schwinger–Dyson equation for the fermion propagator in the ladder approximation. In [2, 3], nonzero solutions were obtained for the mass function and hence for the dynamic fermion mass, and the following alternative conclusions were drawn:

(i) the fermion vacuum in QED_3 can have a complex structure; moreover, each mass has its own vacuum and propagator, i.e.,

$$G_n(x-x') = \langle 0,n | T(\psi(x)\bar{\psi}(x')) | n,0 \rangle ; \tag{1}$$

(ii) the vacuum structure can be simple, and different vacuum structures correspond to different Lagrangians.

The first case describes spontaneous chiral symmetry breaking, and the second case characterizes dynamic symmetry breaking. In this work, the problem of choice between the indicated alternatives in QED_3 is solved based on the ladder approximation for the Bethe–Salpeter wave function of the Goldstone boson and the Landau gauge for the photon propagator.

Statements (i) and (ii) can be illustrated as follows. The term that breaks the chiral invariance in the Lagrangian has the form

$$L_m = m_0(\Lambda)\bar{\psi}(x)\psi(x) , \tag{2}$$

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where Λ is the cutoff parameter and $m_0(\Lambda)$ is the bare fermion mass (it is assumed that $\lim\limits_{\Lambda\to\infty} m_0(\Lambda)=0$). From here, however, it does not follow that $L_m \to 0$ when $\Lambda\to\infty$, since the operator $\bar{\psi}(x)\psi(x)$ can be poorly defined. Therefore, the situation is possible when L_m does not vanish for $m_0(\Lambda)\to 0$, that is, the chiral symmetry can be broken even when $m_0 \to 0$. To obtain an exact answer to the question whether or not this is the case for the examined model, we can, on the one hand, calculate directly the matrix elements of the operator $\bar{\psi}(x)\psi(x)$ and to estimate their behavior for $\Lambda\to\infty$, and on the other hand, we can test the model for the presence of particles with zero mass.

In [4] the choice between cases (i) and (ii) was made by consideration of the effective potential for the examined model. In the present work, the solution of this problem is supplemented by calculation of the Bethe–Salpeter boson wave function $\chi(q,p)$ (here $q_\mu = p_{1\mu} - p_{2\mu}$, $p_\mu = \frac{p_{1\mu}+p_{2\mu}}{2}$, and p_1 and p_2 are the momenta of the fermion and antifermion whose bound state forms this boson). In addition, we consider here the behavior of the compound operator $\bar{\psi}(x)\psi(x)$, in particular, calculate the matrix element $\langle\bar{\psi}(x)\psi(x)\rangle_0$.

2. Considering that the number of axial vector vertices is doubled in QED$_3$ compared to QED$_4$ ($\Gamma_{5\mu}(p_2,p_1)$ and $\Gamma_{3\mu}(p_2,p_1)$) and assuming that axial currents are retained in accordance with [5, 6], for the photon propagator in the ladder approximation with the Landau gauge we obtain the following relationship between the function $\chi(q=0,p)=\chi(p)$ and the mass function $M(p)$:

$$\chi(p)=\frac{1}{f}\frac{M(p)}{p^2+m^2}, \tag{3}$$

where f is a constant related to the boson decay constant, and m is taken to mean the dynamic fermion mass. The equation for the function $\chi(p)$ has the form

$$\left(p^2+m^2\right)\chi(p)=\frac{2e_0^2}{(2\pi)^3}\int\chi(k)\frac{d^3k}{l^2}. \tag{4}$$

Here e_0^2 is the dimensional coupling constant in QED$_3$, $l=p-k$, and p and k are the three-dimensional vectors considered in the Euclidean region. Below we consider approximate and exact variants of solution of Eq. (4) with their subsequent analysis.

a) *Approximate variant*. We now express the kernel $1/l^2$ in the form

$$\frac{1}{l^2}=\frac{\theta(p-k)}{p^2}\sum_{n=1}^{\infty}C_{n-1}^1(\cos\vartheta)\left(\frac{k}{p}\right)^{n-1}+\frac{\theta(k-p)}{k^2}\sum_{n=1}^{\infty}C_{n-1}^1(\cos\vartheta)\left(\frac{p}{k}\right)^{n-1},$$

where θ is the Heaviside function, p and k are the moduli of the corresponding vectors, ϑ is the angle between them, and $C_{n-1}^1(\cos\vartheta)=\frac{\sin n\vartheta}{\sin\vartheta}$ are the Gegenbauer polynomials.

Substituting the above expansion into Eq. (4) and considering the first non-vanishing terms, we obtain

$$\left(p^2+m^2\right)\chi(p)=\frac{e_0^2}{\pi^2p^2}\int_0^p\chi(k)k^2dk+\frac{e_0^2}{\pi^2}\int_p^\infty\chi(k)dk. \tag{5}$$

In the right side of Eq. (5), we have neglected the mass term in comparison with p^2, considered the lower limit of integration m, and proceeded to the differential equation

$$\frac{p^3 d^2\chi}{dp^2} + 7p^2\frac{d\chi}{dp} + \left(8p + \frac{2e_0^2}{\pi^2}\right)\chi = 0 \tag{6}$$

with boundary conditions

$$\left(\left(p^2\right)\chi(p)\right)'_{p=m} = 0, \quad \lim_{p\to\infty}\frac{\left(\left(p^4\right)\chi(p)\right)'}{p} = 0. \tag{7}$$

Equation (6) with conditions (7) has a nonzero solution. Indeed, the general solution of Eq. (6) has the form

$$\chi(p) = p^{-3}\left(c_1 J_2\left(\alpha_p\right) + c_2 N_2(\alpha_p)\right), \tag{8}$$

where J_2 and N_2 are the Bessel and Neumann functions of the corresponding order, and $\alpha_p = \left(8e_0^2/\pi^2 p\right)^{1/2}$.

The second equation of conditions (7) yields $c_2 = 0$, and from the first condition, we obtain

$$\left(\left(p^2\right)\chi(p)\right)'_{p=m} = -\frac{c_1\alpha_m}{2p^2}J_1(\alpha_m) = 0 \tag{9}$$

$\left(\alpha_m = \left(8e_0^2/\pi^2 m\right)^{1/2}\right)$, from which it follows that nonzero value for c_1 is possible only under condition that

$$J_1(\alpha_m) = 0, \tag{10}$$

which coincides with relation (12) from [1] determining the dynamic fermion mass. We have

$$m = m_n = \frac{8e_0^2}{\pi^2 j_{1,n}^2}, \tag{11}$$

where $\alpha_m = j_{1,n}$ are roots of the function $J_1(x)$ of the order n (except zero). It is well known that this function has infinite number of simple roots, and all of them are real and positive [7].

To calculate the constant c_1, we take advantage of equality (3); as a result, we obtain

$$c_1 = \frac{m^2}{f\cdot J_2(\alpha_m)}.$$

Thus, for the Bethe–Salpeter wave function of the Goldstone boson, we have

$$\chi(p) = \frac{m^2\cdot J_2(\alpha_p)}{f\cdot p^3 J_2(\alpha_m)}. \tag{12}$$

It can be easily seen that the expression obtained follows from relations (3) of the present work and (14) of work [1]. We note also that $J_2(\alpha_m) \neq 0$ in equality (12), since the roots of functions $J_1(\alpha_m)$ and $J_2(\alpha_m)$ do not coincide [7].

b) *Exact variant.* As indicated in [1], Eq. (4) in the Euclidean region was considered for the first time by V. A. Fok (see [8]). The solutions of this equation possessing the s-symmetry have the form

$$\chi(p) = C\left(p^2 + m^2\right)^{-2} \frac{\sin n\gamma}{\sin\gamma}, \tag{13}$$

where

$$\gamma = \arccos\frac{m^2 - p^2}{m^2 + p^2}.$$

The constant C here cannot be determined from Eq. (4) due to its linearity; its determination calls for additional conditions. According to [1], for this constant we have

$$C = \frac{2m^3}{f}\frac{1}{\sin\left(n\pi/2\right)}, \tag{14}$$

where only odd values should be taken for n. We note also that solutions (13) appear possible only for the dynamic mass

$$m = \frac{e_0^2}{4\pi n}. \tag{15}$$

The presence of the Goldstone boson is in agreement with the retention of the axial currents (the divergence of the matrix elements of these currents is equal to zero). In this case, the formal chiral symmetry of the Lagrangian appears the actual symmetry of the model spontaneously broken by solutions (14), (17), and (19) of [1]. Thus, the conclusion about the complex structure of the fermion vacuum in QED_3 (statement (i)) can be drawn here based on the presence of the Goldstone boson.

3. We now consider the determination of the operator $\bar{\psi}(x)\psi(x)$ for the model, in particular, find the chiral condensate

$$\langle\bar{\psi}\psi\rangle_{0,n} = \frac{1}{4}\mathrm{Tr}\,G_n(x)\big|_{x=0} = -\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{Tr}\frac{\hat{k} + iM(k)}{k^2 + M^2(k)}d^3k \tag{16}$$

that plays the role of the order parameter. It can be seen that in the examined approximation, the chiral condensate is related to the Bethe–Salpeter wave function by the expression

$$\langle\bar{\psi}\psi\rangle_{0,n} = -\frac{if}{(2\pi)^3}\int \chi(k)d^3k. \tag{17}$$

Thus, the condensate represents the Bethe–Salpeter wave function for the Goldstone boson at the origin of coordinates multiplied by the probability of its decay. It can be calculated using both results of approximate calculations (12) and exact calculations (13).

a) *Approximate calculations.* Proceeding in equality (17) to the Euclidean variables, introducing infrared cutting m_n, neglecting in the denominator by the function $M^2(k)$ in comparison with k^2, and integrating over the angular variable, we obtain

$$\langle \bar{\psi}\psi \rangle_{0,n} = \frac{m_n^2}{\pi^2 J_2(\alpha_m)} \int_{m_n}^{\infty} \frac{J_2\left(\alpha k^{-1/2}\right)}{k} dk = \frac{m_n^2}{\pi^2 J_2(\alpha_m)} \int_0^1 x^{-1} J_2(\alpha_m x) dx \,, \tag{18}$$

where $\alpha = \left(\frac{8e_0^2}{\pi^2}\right)^{1/2}$. Calculation of the last integral yields

$$\int_0^1 x^{-1} J_2(\alpha_m x) dx = \frac{1}{2} - \alpha_m^{-1} J_1(\alpha_m) \,.$$

Taking advantage of Eq. (10), we have

$$\langle \bar{\psi}\psi \rangle_{0,n} = \frac{m_n^2}{2\pi^2 J_2(\alpha_m)} \,. \tag{19}$$

Using Eq. (11), we obtain for $\langle \bar{\psi}\psi \rangle_{0,1}$:

$$\langle \bar{\psi}\psi \rangle_{0,1} = 5.12 \cdot 10^{-4} e_0^4 \,. \tag{20}$$

The reasons why we consider the case $n = 1$ here and in section 3*b* are discussed in section 4.

The chiral condensate and the mass function are related by the formula

$$M_n(p) = \frac{\pi^2}{2} \langle \bar{\psi}\psi \rangle_{0,n} \, p^{-1} J_2(\alpha_p) \,, \tag{21}$$

derived in [9] using the procedure of expansion of the operator product. In this work, Eq. (21) was derived as a result of direct solution of the boundary problem for Eq. (6) with conditions (7).

b) *Exact calculations.* To determine the chiral condensate using the exact Bethe–Salpeter wave function, we take advantage of representations (12), (13), and (16) for $n = 1$ which, as demonstrated below (section 4) and follows from [1], corresponds to the actual situation. Calculations in the Euclidean region yield

$$\langle \bar{\psi}\psi \rangle_{0,1} = \frac{m_1^2}{4\pi} = 7.74 \cdot 10^{-4} e_0^4 \,. \tag{22}$$

Considering that Eq. (22) is exact, we can see that the quantitative difference between Eqs. (20) and (22) is 34%. Thus, both expressions yield values of the same orders of magnitude and, considering that the purely technical approximations are rough, the agreement of the results obtained from Eqs. (20) and (22) can be considered satisfactory.

Finite values of expressions (18) and (20) for the chiral condensate allow us to think that other matrix elements of the compound operator $\bar{\psi}(x)\psi(x)$ for our model will not require additional determination, and hence term (2) will disappear for any arbitrary law according to which $m_0(\Lambda)$ tends to zero. Anyway, the Lagrangian will be chiral invariant for $m_0 = 0$. In other words, the behavior of the chiral condensate for $\Lambda \to \infty$ in this model corresponds to case (i).

4. Thus, in the present work it has been demonstrated that the dynamic mass of fermion in QED_3 in the ladder approximation arises due to spontaneous rather than dynamic chiral symmetry breaking [2, 3] (case (i)). We now analyze in more detail the physical meaning of the results obtained. First of all, we indicate the physically special case of $n = 1$. It possesses the greatest fermion mass and, hence, the least time of massive phase formation. To solve Eq. (4),

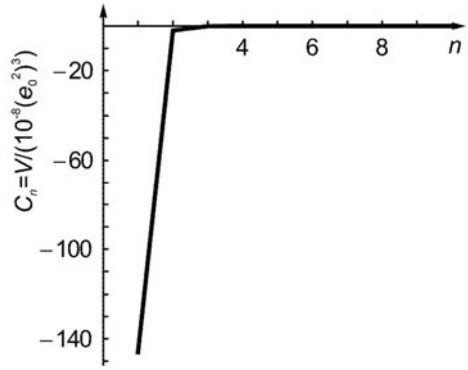

Fig. 1. Dependence of the effective potential on the serial channel number.

the fermion mass corresponding to the channel with the given n value must be used. We note that in [1] it was demonstrated that solutions of the Schwinger–Dyson equation for the mass function for $n > 1$ are artifacts of linearization of the given equation.

Other reasons can be given if we consider the analogy between the examined quantum field problem and the one-particle problem of relativistic quantum mechanics on particle motion in the field of a central force [6]. Indeed, analyzing Eq. (12) for the Bethe–Salpeter wave function of the Goldstone boson, we see that for $n = 1$, it has no zeros in the region $p \geq m$ and hence, the boson state with $M_0^2 = 0$ (M_0 is the boson mass) can be considered here as the ground state. For $n > 1$, the examined functions start to oscillate, and with increasing n, their frequency also increases. This circumstance allows us to consider the boson state with $M_0^2 = 0$ as excited, and all other states must have $M_0^2 < 0$, that is, be tachyon states. As pointed out in [6], in the one-loop approximation of quantum field theory, energy ε with $\operatorname{Im} \varepsilon > 0$ corresponds to the vacuum state with $M_0^2 < 0$, that is, the vacuum state is unstable in this case. Unlike QED_4, in QED_3 there is no critical parameter at which the bound state corresponding to the Goldstone boson is formed from the fermion-antifermion pair. In other words, the above-mentioned boson in QED_3 is produced for any arbitrary value of the coupling constant.

Thus, only the solution with $n = 1$ yields the stable vacuum for the Goldstone bosons; other solutions should be considered as the artifact of the employed approximation. The same conclusion can be drawn from an analysis of Eq. (13), except for the circumstance that here it refers to the entire interval of momenta rather than to the region with $p \geq m$, as in the previous case.

Let us now compare the results of this work with those we obtained by the effective potential method in [4], where we concluded that the structure of vacuum of the massive phase is complex in QED_3. Results of this work and [1] demonstrate that this conclusion, leaning upon the correct factological base, is not true. At the same time, if we present graphically the results obtained in [4] (see Fig. 1, where the serial channel number is considered as a continuous variable), we can see that nonzero values of the effective potential in QED_3 in the employed approximation for $n > 1$ should most likely be interpreted as *noise* around the zero value. In other words, results obtained in [4] can also be considered as demonstrating the simplicity of vacuum of the massive phase.

Thus, in the present work we have demonstrated that the dynamic fermion mass in QED_3 in the ladder approximation results from the spontaneous rather than dynamic chiral symmetry breaking [2, 3], that is, case (i) is realized for this model. The fermion vacuum, as expected in this situation, has complex character and incorporates the massless and massive phases. On the other hand, the vacuum of the massive phase is simple and corresponds to the greatest mass of the obtained spectrum of masses: solutions of the Bethe–Salpeter equation for the wave function of the Goldstone boson that refer to other masses correspond to the excited states and yield an unstable ground condition for this boson.

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